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THE MESSENGER OF MATHEMATICS,

EDITED BY

W. ALLEN WHITWORTH, M.A.,

FELLOW OF ST. JOHN'S COLLEGE, CAMBRIDGE.

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ERRATUM. p. 25, fifth line from bottom, *for* fig. 8 *read* fig. 3.*

MESSENGER OF MATHEMATICS.

A GEOMETRICAL CONSTRUCTION RELATING TO IMAGINARY QUANTITIES.

By Professor *Cayley*.

LET A, B, C be given imaginary quantities, and let it be required to construct the roots of the quadric equation

$$\frac{1}{X-A}+\frac{1}{X-B}+\frac{1}{X-C}=0.$$

The equation is

$$(X-B)(X-C)+(X-C)(X-A)+(X-A)(X-B)=0,$$

that is

$$3X^2-2(A+B+C)X+BC+CA+AB=0,$$

and we have therefore

$$3X-(A+B+C)=\pm\sqrt{\{(A+B+C)^2-3(BC+CA+AB)\}},$$
$$=\pm\sqrt{\{A^2+B^2+C^2-BC-CA-AB\}};$$

or as this may be written

$$X=\tfrac{1}{3}(A+B+C)\pm\sqrt{\{\tfrac{1}{3}(A+B\omega+C\omega^2).\tfrac{1}{3}(A+B\omega^2+C\omega)\}},$$

where ω is an imaginary cube root of unity,

$$=\cos 120^\circ+i\sin 120^\circ \text{ suppose.}$$

Taking an arbitrary point O as the origin, let the imaginary quantity $A=\alpha+\alpha' i$ suppose be represented by the point A, coordinates α and α'; and in like manner the imaginary quantities B and C by the points B and C respectively.

Then $B\omega$, $B\omega^2$ are represented by points B_1, B_2, obtained by rotating the point B about the origin through angles of 120° and 240° degrees respectively; $C\omega^2$, $C\omega$ are represented by points C_1, C_2 obtained by rotating the point C about the origin through angles of 240° and 480° (= 120°) respectively: and $\frac{1}{3}(A+B+C)$, $\frac{1}{3}(A+B\omega+C\omega^2)$, $\frac{1}{3}(A+B\omega+C\omega^2)$ are

represented by the points G, G_1, G_2 which are the C. G's of the triangles ABC, AB_1C_1, AB_2C_2 respectively. The formula therefore is

$$X = OG \pm \sqrt{(OG_1 . OG_2)},$$

where, if a, a' are the coordinates of G, then OG is written to denote the imaginary quantity $a + a'i$; and the like as regards OG_1, OG. Taking $\sqrt{(OG_1 . OG_2)} = OH$, we then have H a point such that the distance OH from the origin is = geometric mean of the distances OG_1, OG_2; and that the radial direction* of the distance OH bisects the radial directions of the distances OG_1, OG_2 respectively. Finally, measuring off from G in the radial direction OH, and in the opposite radial direction, the distances GX', GX'' each $= OH$; we have the two points X', X'' representing the two roots X.

The construction is somewhat simplified if we take for the origin the point G; for then $OG = 0$, and we have $X = \pm \sqrt{(GG_1 . GG_2)}$, so that the points X', X'' are in fact the point H, and the opposite point in regard to G.

The theory of the more general equation

$$\frac{p}{X-A} + \frac{q}{X-B} + \frac{r}{X-C} = 0,$$

(p, q, r real) is somewhat similar, but the construction is less simple; we have

$$(p+q+r) X^2 - \{(q+r) A + (r+p) B + (p+q) C\} X + pBC + qCA + rAB = 0.$$

Writing herein $q+r, r+p, p+q = l, m, n$, the equation becomes

$$(l+m+n) X^2 - 2 (lA + mB + nC) X + (-l+m+n) BC + (l-m+n) CA + (l+m-n) BC = 0,$$

that is

$$\{(l+m+n) X - lA - mB - nC\}^2 = (lA + mB + nC)^2 + \{l^2 - (m+n)^2\} BC + \{m^2 - (n+l)^2\} CA + \{n^2 - (l+m)^2\} AB.$$

Here the right-hand side is

$$= l^2A^2 + m^2B^2 + n^2C^2 + (l^2 - m^2 - n^2) BC + (-l^2 + m^2 - n^2) CA + (-l^2 - m^2 + n^2) AB,$$

* Radial direction, is I think a convenient expression for the direction of a line considered as drawn as a radius of a circle from the centre, and not as a diameter in two opposite radial directions.

which is

$$= -l^2(C-A)(A-B) - m^2(A-B)(B-C) - n^2(C-A)(A-B),$$

and consequently a product of two linear factors; these in fact are

$$\frac{1}{l}\{l^2A + \tfrac{1}{2}(-l^2 - m^2 + n^2 \pm \sqrt{\Delta})B + \tfrac{1}{2}(-l^2 + m^2 - n^2 \mp \sqrt{\Delta})C\},$$

where

$$\Delta = l^4 + m^4 + n^4 - 2m^2n^2 - 2n^2l^2 - 2l^2m^2.$$

It is to be observed that Δ, $= (l^2 - m^2 - n^2)^2 - 4m^2n^2$, is negative; hence calling the factors $fA+gB+hC$, $f'A+g'B+h'C$ respectively, the coefficients f, g, h, and f', g', h' are consequently imaginary; and moreover $f+g+h=0$, $f'+g'+h'=0$.

The values of X thus are

$$(l+m+n)X = lA+mB+nC \pm \sqrt{(fA+gB+hC)(f'A+g'B+h'C)},$$

and then passing to the geometrical representation, we have $\frac{lA+mB+nC}{l+m+n}$ represented by the point which is the C. G. of weights l, m, n at the points A, B, C respectively; on account of the imaginary values of the coefficients the construction is not immediately applicable to the factors

$$fA+gB+hC, \quad f'A+g'B+h'C;$$

but a construction such as was used for the factors

$$A+\omega B+\omega^2 C, \quad A+\omega^2 B+\omega C$$

might be found without difficulty.

ON FREE MOTION UNDER THE ACTION OF SEVERAL CENTRAL FORCES.

By *Arthur Hill Curtis, LL.D.*

WHEN a particle describes a curve freely under the action of any number of forces, the equations of motion can be reduced to the two following:

$$v^2 = F_1(\tfrac{1}{2}c_1) + F_2(\tfrac{1}{2}c_2) + \&c. \ldots\ldots\ldots\ldots (1),$$

$$vdv = -F_1dr_1 - F_2dr_2 - \&c. \ldots\ldots\ldots\ldots (2),$$

F_1, F_2, &c. being forces acting along r_1, r_2, &c., and tending to diminish them, while c_1, c_2, &c. are the chords of curvature coinciding in direction with these lines respectively, and dr_1, dr_2, &c. are the projections on them of the element of the curvilinear path of the particle.

These equations express the two facts, that the centrifugal force is equal and opposite to the normal component of the resultant of F_1, F_2, &c.; and that the acceleration along the tangent is due to the tangential component of the same. We may for shortness write these equations thus:

$$v^2 = \Sigma F(\tfrac{1}{2}c) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3),$$

$$vdv = -\Sigma F dr \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4),$$

differentiating (3), we obtain

$$\Sigma(Fdc + cdF) = 4vdv = -4\Sigma F dr;$$

therefore

$$\Sigma\{F(dc + 4dr) + cdF\} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5).$$

Now, if ϕ_1, ϕ_2, &c. denote the forces respectively codirectional with F_1, F_2, &c., under which *singly* the given curve would be described, we must have, as particular cases of (5),

$$\phi_1(dc_1 + 4dr_1) + c_1 d\phi_1 = 0,$$

$$\phi_2(dc_2 + 4dr_2) + c_2 d\phi_2 = 0,$$

$$\&c \qquad \&c.;$$

therefore

$$dc_1 + 4dr_1 = -\frac{c_1 d\phi_1}{\phi_1},$$

$$\&c. \qquad \&c.;$$

therefore from (5)

$$\Sigma c\left(dF - \frac{Fd\phi}{\phi}\right) = 0,$$

or

$$\Sigma c\phi d\left(\frac{F}{\phi}\right) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6),$$

which is equivalent to

$$c_1\phi_1 d\left(\frac{F_1}{\phi_1}\right) + c_2\phi_2 d\left(\frac{F_2}{\phi_2}\right) + \&c. = 0,$$

a condition which must be satisfied *at every point on the orbit.*

The above equation may be satisfied in an infinite variety of ways; one very obvious solution is obtained by assuming $F_1 = K_1\phi_1$, $F_2 = K_2\phi_2$, &c., &c.; hence we infer that, if an

orbit be such that it can be described by a particle under the action of *each* of a number of forces acting towards fixed or moveable centres, including the case where some of the centres may be at an infinite distance, this orbit can also be described by a particle acted on by these forces acting *simultaneously*, each being multiplied by an arbitrary constant.

If v_0 be the velocity of the particle at the initial position, which may be supposed to be the same whether the forces act *singly* or *simultaneously*, while the direction of the initial velocity, being the tangent to the given orbit at this point, will also be invariable, then

$$V_0^2 = \Sigma F(\tfrac{1}{2}c) = \Sigma K\phi(\tfrac{1}{2}c) = K_1(V_1)_0^2 + K_2(V_2)_0^2 + \&c.,$$

which gives the required initial velocity when the forces all act simultaneously in terms of those required when each of them acts singly.

This includes the two cases proved *indirectly* by Walton in his "Examples," viz.:

1°. With a proper initial velocity, a particle will describe a given ellipse under the simultaneous action of two forces directed towards the two foci; each varying according to the inverse square of the distance.

2°. With a proper initial velocity, a particle will describe a given ellipse under the simultaneous action of three forces directed towards the two foci and the centre; the two first varying inversely as the square of the distance, and the third directly as the distance.

The following problem, proposed among the Senate-House Problems in 1878, is also a very particular case of the above:

If a body describe a parabola under the action of two forces, one constant and parallel to the axis, the other a force directed towards the focus, the latter must vary inversely as the square of the distance. For the condition in this case is

$$c_1\phi_1 d\left(\frac{F_1}{\phi_1}\right) + c_2\phi_2 d\left(\frac{F_2}{\phi_1}\right) = 0,$$

where F_2 and ϕ_2 are both constants; therefore

$$c_1\phi_1 d\left(\frac{F_1}{\phi_1}\right) = 0;$$

therefore

$$\frac{F_1}{\phi_1} = K,$$

and, as $\phi_1 = \frac{\mu}{r_1^2}$, $\qquad F_1 = \frac{K\mu}{r_1^2}$.

Professor MacCullagh and Sir W. R. Hamilton proved that a particle will describe an ellipse under the action of a force directed towards any point within it, if the force vary directly as the distance of the particle from the centre of force and inversely as the cube of the particle from the polar (with regard to the ellipse) of the centre of force; hence, a particle with a certain initial velocity will describe a given ellipse under the simultaneous action of a number of forces directed towards any number of points situated within it, if each of these forces is directly proportional to the distance of the particle from the corresponding centre of force and inversely proportional to the cube of the distance of the particle from the polar (with regard to the ellipse) of the centre of force.

These are, however, only *very special applications* of equation (6):—we may assign *any* law for *all but one* of the functions $F_1, F_2, \ldots F_n$, and the condition (6) will determine the law of the remaining one.

Let us, for example, consider the laws of forces directed towards the two foci of an elliptic orbit described under their combined action.

The condition is

$$c_1\phi_1 d\left(\frac{F_1}{\phi_1}\right) + c_2\phi_2 d\left(\frac{F_2}{\phi_2}\right) = 0,$$

or, as $\qquad \phi_1 = \frac{\mu_1}{r_1^2}$, $\phi_2 = \frac{\mu_2}{r_2^2}$, and $c_1 = c_2$,

$$\frac{1}{r_1^2}\frac{d}{dr_1}(F_1 r_1^2)\,dr_1 + \frac{1}{r_2^2}\frac{d}{dr_2}(F_2 r_2^2)\,dr_2 = 0,$$

but, as $r_1 + r_2 = 2a$, $dr_2 = -dr_1$, and therefore

$$\frac{1}{r_1^2}\frac{d}{dr_1}(F_1 r_1^2) - \frac{1}{r_2^2}\frac{d}{dr_2}(F_2 r_2^2) = 0,$$

with the condition $r_1 + r_2 = 2a$.

The most general solution of this equation appears to be

$$\frac{1}{r^2}\frac{d}{dr}(Fr^2) = f(r) + f(2a - r),$$

(f denoting an arbitrary function)

or $\qquad F = \frac{1}{r^2}\int r^2 \{f(r) + f(2a - r)\}\,dr.$

If we assign to f *any* form, a corresponding form for F, and therefore for F_1 and F_2 will result: *e.g.*

1°. If f be defined by $f(r) = K$, then

$$F = \frac{1}{r^3}\int 2Kr^2 dr = \frac{1}{r^3}\left(\tfrac{2}{3}Kr^3 + \mu\right)$$

$$= \left(\tfrac{2}{3}K\right)r + \frac{\mu}{r^3};$$

therefore $$F_1 = \lambda r_1 + \frac{\mu_1}{r_1^3}, \quad F_2 = \lambda r_2 + \frac{\mu_2}{r_2^3}$$

2°. If f be defined by $f(r) = Kr + K'$, then

$$F = \frac{1}{r^3}\int r^2\{Kr + K' + K(2a - r) + K'\}\, dr$$

$$= \frac{1}{r^3}\int r^2 2\,(Ka + K')\, dr = \lambda r + \frac{\mu}{r^3},$$

the same law as in 1°.

3°. If f be defined by $f(r) = Kr^2 + K'r + K''$, then

$$F = \frac{1}{r^3}\int r^2\left[K\{r^2 + (2a - r)^2\} + K'(r + 2a - r) + 2K''\right] dr,$$

or $$F = \frac{1}{r^3}\int r^2\{2K(r^2 - 2ar + 2a^2) + 2K'a + K''\}\, dr$$

$$= \frac{1}{r^3}\int r^2\{2K(r^2 - 2ar) + 3l\}\, dr$$

$$= \frac{1}{r^3}\left[2K\left\{\tfrac{1}{5}r^5 - \left(\tfrac{1}{2}a\right)r^4\right\} + lr^3 + \mu\right]$$

$$= 2K\left\{\tfrac{1}{5}r^5 - \tfrac{1}{2}(ar^4)\right\} + lr + \frac{\mu}{r^3},$$

K, l, and μ being arbitrary constants, of which K and l *must be the same* in the expressions for F_1 and F_2, *though* μ *need not be;* for as we may add to F_1, or F_2, or both, any term which satisfies the condition

$$\frac{1}{r^2}\frac{d}{dr}(Fr^2) = 0,$$

viz. the term $\frac{V}{r^{\varkappa}}$, where V is any arbitrary constant, we may put

$$F_1 = K\left(\tfrac{2}{5}r_1^5 - ar_1^4 + \lambda r_1 + \frac{\mu_1}{r_1^2}\right),$$

$$F_2 = K\left(\tfrac{2}{5}r_2^5 - ar_2^4 + \lambda r_2 + \frac{\mu_2}{r_2^2}\right).$$

When F_1 and F_2 are found by assigning to f any form, we may combine with them a force directed towards the centre and varying directly as the distance, or any forces directed towards fixed points within the ellipse and varying according to Professor Mac Cullagh's law, and the same orbit will be describable under their simultaneous action.

The analogy between the orbit described under the action of a central force and the form of the curve assumed by a string acted on by a central force, from which several elegant results have been deduced by Professor Townsend, of Trinity College, Dublin, naturally suggests that a condition similar to, if not identical with, the condition (6) must hold between a system of forces, under which a string will be in equilibrium in a given form; in fact, the equations of equilibrium of a string so circumstanced may be written, with the usual notation,

$$\frac{T}{m} = F_1\tfrac{1}{2}c_1 + F_2\tfrac{1}{2}c_2 + \&c. = \Sigma F\tfrac{1}{2}c,$$

$$-dT = F_1 m dr_1 + F_2 m dr_2 + \&c. = \Sigma F m dr;$$

hence, $$-2\Sigma F m dr = d\Sigma m F c$$

$$= \Sigma\{mcdF + Fd(mc)\},$$

or $$\Sigma[mcdF + F\{d(mc) + 2mdr\}] = 0 \ldots\ldots\ldots\ldots(7);$$

but, if ϕ_1, ϕ_2, &c. be forces codirectional, respectively, with F_1, F_2, &c., under which *singly* the string would retain the same form, the value of m at any assumed point on the string being the same *in all cases*, we must have

$$mc_1 d\phi_1 + \phi_1\{d(mc_1) + 2mdr_1\} = 0,$$

$$mc_2 d\phi_2 + \phi_2\{d(mc_2) + 2mdr_2\} = 0,$$

$$\&c. \qquad\qquad \&c.$$

therefore $$d(mc_1) + 2mdr_1 = -\frac{mc_1 d\phi_1}{\phi_1};$$

&c. &c.

and, consequently, from equation (7)

$$\Sigma mc\left(dF - \frac{Fd\phi}{\phi}\right) = 0,$$

or $$\Sigma mc\phi d\left(\frac{F}{\phi}\right) = 0,$$

a condition which must be fulfilled *at every point* on the curve, the quantity m, though it may be variable from point to point, being a common factor may be omitted, and the condition becomes

$$\Sigma c\phi d\left(\frac{F}{\phi}\right) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (8),$$

$$c_1\phi_1 d\left(\frac{F_1}{\phi_1}\right) + c_2\phi_2 d\left(\frac{F_2}{\phi_2}\right) + \&c. = 0,$$

the tension at each extremity being given by the relation

$$T_0 = \{F_1 \tfrac{1}{2}(c_1) + F_2 \tfrac{1}{2}(c_2) + \&c.\}_0.$$

One consequence of the condition (8) is that, if a string be in equilibrium under the action of each of a number of systems of forces, it will also be in equilibrium under the action of the system compounded of them, each being multiplied by an arbitrary constant, for the assumptions

$$F_1 = K_1\phi_1, \quad F_2 = K_2\phi_2, \text{ \&c.},$$

will satisfy the condition (8), and the tension at each extremity will be given by the relation

$$T_0 = K_1\phi_1 \tfrac{1}{2}(c_1) + K_2\phi_2 \tfrac{1}{2}(c_2) + \&c.,$$

$$= K_1(T_1)_0 + K_2(T_2)_0 + \&c.;$$

we may, however, as before, give to each but one of the functions F_1, F_2, &c. any arbitrary form, and the condition (8) will determine the remaining one.

A number of consequences—analogous to those already established in the case of an orbit, may be deduced; it will be sufficient to discuss the case in which a *homogeneous* string assumes the form of an ellipse under forces acting from the two foci and the centre—either singly or conjointly.

1°. If the only force be one directed from one focus, then, denoting by r_1, r_2 the distances of any point on the ellipse from the two foci, and by p_1, p_2 the perpendiculars let fall from the two foci on the tangent at the point

$$\frac{r_2}{r_1} = \frac{p_2}{p_1} \text{ and } p_1 p_2 = b^2;$$

therefore $$\frac{1}{p_1^2} = \frac{1}{b^2}\frac{r_2}{r_1} \text{ and } \frac{1}{p_1} = \frac{1}{b}\sqrt{\left(\frac{r_2}{r_1}\right)},$$

but $$-\phi_1 dr = dT_1 = d\frac{C}{p_1} = \frac{C}{b} d\sqrt{\left(\frac{r_2}{r_1}\right)} = \frac{C}{b}\frac{r_1 dr_2 - r_2 dr_1}{2r_1^2\sqrt{\left(\frac{r_2}{r_1}\right)}};$$

or, as $$dr_2 = -dr_1,$$

$$\phi_1 = \frac{C}{b}\frac{r_1 + r_2}{2r_1^2\sqrt{\left(\frac{r_2}{r_1}\right)}} = \frac{C}{b}\frac{2a}{2r_1^2\sqrt{\left(\frac{r_2}{r_1}\right)}} = \frac{\mu_1}{r_1^{\frac{5}{2}} r_2^{\frac{1}{2}}}.$$

2°. If two forces act, one from each focus, then the condition (8) becomes

$$c_1\phi_1 d\left(\frac{F_1}{\phi_1}\right) + c_2\phi_2 d\left(\frac{F_2}{\phi_2}\right) = 0,$$

or, as $c_1 = c_2$,

$$\phi_1 \frac{d}{dr_1}\left(\frac{F_1}{\phi_1}\right) dr_1 + \phi_2 \frac{d}{dr_2}\left(\frac{F_2}{\phi_2}\right) dr_2 = 0,$$

or, as $dr_2 = -dr_1$,

$$\phi_1 \frac{d}{dr_1}\left(\frac{F_1}{\phi_1}\right) - \phi_2 \frac{d}{dr_2}\left(\frac{F_2}{\phi_2}\right) = 0,$$

but $$\phi_1 = \frac{\mu_1}{r_1^{\frac{5}{2}} r_2^{\frac{1}{2}}}, \text{ and } \phi_2 = \frac{\mu_2}{r_2^{\frac{5}{2}} r_1^{\frac{1}{2}}};$$

therefore

$$\frac{1}{\sqrt{(r_1 r_2)}}\left\{\frac{1}{r_1}\frac{d}{dr_1}(F_1 r_1^{\frac{1}{2}} r_2^{\frac{1}{2}}) - \frac{1}{r_2}\frac{d}{dr_2}(F_2 r_2^{\frac{1}{2}} r_1^{\frac{1}{2}})\right\} = 0.$$

The most general solution of this equation appears to be

$$\frac{1}{r_1}\frac{d}{dr_1}(F_1 r_1^{\frac{1}{2}} r_2^{\frac{1}{2}}) = f(r_1) + f(2a - r_1),$$

or $$F_1 = \frac{1}{r_1^{\frac{1}{2}} r_2^{\frac{1}{2}}} \int \{f(r_1) + f(2a - r_1)\}\, r_1 dr_1,$$

f denoting any arbitrary function; *e.g.* let

$$f(r) = Kr^2 + K'r + K'',$$

then

$$F_1 = \frac{1}{r_1^{\frac{1}{2}} r_2^{\frac{1}{2}}} \int \{Kr_1^2 + K'r_1 + K'' + K(2a - r_1)^2 + K'(2a - r_1) + K''\}\, r_1 dr_1;$$

therefore

$$F_1 = \frac{1}{r_1^{\frac{1}{2}} r_2^{\frac{1}{2}}} \int \{K(2r_1^2 - 4ar_1) + (4Ka^2 + 2K'a + 2K'')\}\, r_1 dr_1$$

$$= \frac{1}{r_1^{\frac{1}{2}} r_2^{\frac{1}{2}}} \{K(\tfrac{1}{2} r_1^4 - \tfrac{4}{3} a r_1^3) + l r_1^2 + \mu_1\}$$

$$F_1 = \frac{1}{(2a - r_1)^{\frac{1}{2}}} \left\{\tfrac{1}{2} K r_1^{\frac{5}{2}} - \tfrac{4}{3} K a r_1^{\frac{3}{2}} + l r_1^{\frac{1}{2}} + \frac{\mu_1}{r_1^{\frac{1}{2}}}\right\}.$$

Similarly

$$F_2 = \frac{1}{(2a - r_2)^{\frac{1}{2}}} \left\{\tfrac{1}{2} K r_1^{\frac{5}{2}} - \tfrac{4}{3} K a r_2^{\frac{3}{2}} + l r_2^{\frac{1}{2}} + \frac{\mu_1}{r_2^{\frac{1}{2}}}\right\},$$

μ_2 and μ_1 not being necessarily the same, as we may add to F_1 any term which satisfies the equation $\frac{d}{dr_1}(F_1 r_1^{\frac{1}{2}} r_2^{\frac{1}{2}}) = 0$, viz. the term $\frac{v}{r_1^{\frac{1}{2}} r_2^{\frac{1}{2}}}$.

3°. When the elliptic form is that of equilibrium of a homogeneous string acted on by a force directed from its centre, denoting by r_2 the distance of any point on the ellipse from its centre, we have $p_2 (a^2 + b^2 - r_2^2)^{\frac{1}{2}} = ab$; therefore

$$\frac{1}{p_2} = \frac{(a^2 + b^2 - r_2^2)^{\frac{1}{2}}}{ab};$$

therefore $$-\phi_2 dr_2 = d\left(\frac{C'}{p_2}\right)$$

$$= \frac{C'}{ab} \left\{\frac{-r_2}{(a^2 + b^2 - r_2^2)^{\frac{1}{2}}}\right\} dr_2,$$

or $$\phi_2 = \frac{\mu_2 r_2}{(a^2 + b^2 - r_2^2)^{\frac{1}{2}}}.$$

4°. We may, then, by the theorem, combine the forces F_1, F_2 and ϕ_x *as above determined*, and the elliptic form will still be that of equilibrium; but this is not the most general solution, for, as already remarked, we may assign to two of the functions any forms we chose, and by equation (8) determine the remaining function.

TRANSACTIONS OF SOCIETIES.

London Mathematical Society.

Thursday, Dec. 11th, 1879. C. W. Merrifield, Esq., F.R.S., *President*, in the Chair.

The following gentlemen were elected members: Mr. W. Burnside, M.A., Fellow of Pembroke College, Cambridge; Mr. J. R. Harris, M.A., Fellow of Clare College, Cambridge; Dr. W. Jack, Professor of Mathematics in the University of Glasgow; Mr. W. J. Curran Sharp, Ex-Scholar of Trinity College, Dublin; and Prof. W. Woolsey Johnson, St. John's College, Annapolis, Maryland, U.S.A.

The following communications were made to the Society: "Note on a method of obtaining the q formula for the sine amplitude in elliptic functions," Mr. J. W. L. Glaisher, F.R.S.; "Note on a numerical theorem connected with the cubical division of space," the President; "Notes on curvature," Mr. J. J. Walker, M.A.; "A property of a linkage," Mr. A. B. Kempe, B.A.

Thursday, Jan. 8th, 1880. C. W. Merrifield, Esq., F.R.S., *President*, in the Chair. Prof. W. Snow Burnside, M.A., Trinity College, Dublin, was elected a member. Prof. Cayley, F.R.S., read a paper entitled "A theorem in Spherical Trigonometry," and made some remarks upon a posthumous paper of Prof. Clifford's "On the Theory of Distances." The Chairman supplemented his communication of Dec. 11th by a few further remarks. Mr. J. Hammond, M.A., gave an expression for the complementary function in fractional differentiation, upon which a discussion ensued, in which Prof. Cayley, the Chairman, Messrs. Roberts, Glaisher, and Freeman, took part.

Thursday, Feb. 12th, 1880. C. W. Merrifield, Esq., F.R.S., *President*, in the Chair. Mr. David Edwardes was elected a member and subsequently admitted into the Society. The following communications were made to the Society: "Geometrical Notes," Prof. H. J. S. Smith, F.R.S.; "On the reflection of vibrations at the confines of two media between which the transition is gradual, and on the stability or instability of certain fluid motions," Lord Rayleigh, F.R.S.; "The calculus of equivalent statements" (Fourth Note), Mr. Hugh McColl, B.A.

Thursday, March 11th, 1880. C. W. Merrifield, Esq., F.R.S., *President*, in the Chair. Mr. W. J. Curran Sharp was admitted into the Society, and the following gentlemen were elected members: Mr. C. S. Peirce, Johns Hopkins University, Baltimore; Mr. Emory McClintock, Milwaukee, Illinois; Prof. E. B. Seitz, Kirksville, Missouri; and Mr. E. Temperley, M.A., Fellow of Queens' College, Cambridge. Dr. Elling Holst, of Christiania, was present. The following communications were made: Notes on a general method of solving partial differential equations of the first order with several dependent variables, Mr. H. W. Lloyd Tanner, M.A.; "Note on the integral solution of $x^2 - 2Py^2 = -z^2$ or $+2z^2$ in certain cases," Mr. S. Roberts, F.R.S.; "Notes (1) on a geometrical form of Landen's theorem with regard to a hyperbolic arc; (2) on a class of closed ovals whose arcs possess the same property as two Fagnanian arcs of an ellipse," Mr. J. Griffiths, M.A.

R. TUCKER, M.A., *Hon. Sec.*

TANGENTIAL COORDINATES.

By *C. Taylor, M.A.*

1. In connexion with the method of tangential coordinates, it may be worth while to call attention to a general property of the tangents to conics proved by Newton, and to a special case of it afterwards proved by Lambert.

Newton has shewn geometrically, in the fifth section of the first book of the *Principia*, that if a variable tangent to a given conic makes intercepts TP and TQ on two fixed tangents, there exist fixed points A and B on TP and TQ respectively, such that

$$AP.BQ = (TA - TP)(TB - TQ) = \text{a constant}.$$

The relation between TP and TQ is therefore of the form

$$a.TP.TQ + b.TP + c.TQ + d = 0,$$

which is a "tangential equation" to the conic referred to any two fixed tangents.

2. *The Parabola.*

It has been shewn by Lambert that in the special case of the parabola the above relation reduces to

$$\frac{TP}{Tp} + \frac{TQ}{Tq} = 1,$$

where p and q are the points of contact of the fixed tangents (*Messenger* IX. 66, § 5). Conversely, every line PQ which satisfies the above relation is a tangent to the parabola, or the envelope of PQ is a parabola *which makes intercepts equal to Tp and Tq on the fixed lines.*

The following are some applications to known theorems of this property of the parabola, which follows from simple geometrical considerations, and was known long before the introduction of the method of tangential coordinates.

3. *The envelope of the polars of a given point with respect to a system of confocal conics is a parabola, whose directrix passes through their common centre.*

Let S and S' be the common foci of a system of conics in one plane, CT and Ct the intercepts upon their transverse and

conjugate axes made by the polar of a fixed point O with respect to any one of the confocals.

Then if h and k be the distances of O from the conjugate and transverse axes respectively,

$$h.CT = C\alpha^2, \text{ and } k.Ct = \beta^2,$$

where $C\alpha$ and $C\beta$ are the lengths of the transverse and conjugate semiaxes.

Hence $$h.CT - k.Ct = CS^2$$

and the envelope of Tt is therefore a parabola, which touches the axes of the confocals and has C for a point on its directrix.

This parabola makes intercepts CX and CY upon the axes such that

$$CX = \frac{CS^2}{h}, \text{ and } CY = -\frac{CS^2}{k}.$$

Its focus F being the foot of the perpendicular from C to XY, therefore

$$\frac{1}{CF^2} = \frac{1}{CX^2} + \frac{1}{CY^2} = \frac{CO^2}{CS^4},$$

or $$CF.CO = CS^2.$$

Hence also $$CF : CX = h : CO,$$

or CF and CO are equally inclined to the transverse axis.

For the following theorems see Mr. Wolstenholme's article in the *Quarterly Journal of Mathematics* x. 285–8.

4. *If OP be either tangent from the fixed point O to any conic of the system, the normal at P touches the fixed parabola which is the envelope of the polars of O.*

For if OP and the normal at P meet the axes in M, N and m, n respectively, then since

$$\frac{h}{CM} + \frac{k}{Cm} = 1,$$

and $$CM.CN = CS^2 = -Cm.Cn;$$

therefore $$h.CN - k.Cn = CS^2,$$

or the envelope of Nn is the fixed parabola above determined. This amounts to saying that the normal Nn is the polar of O with respect to one of the confocals.

5. *If OP and OQ be the tangents from the fixed point O to any conic of the system, the circle OPQ passes through a second fixed point.*

For PQ and the normals at P and Q form a triangle circumscribing the fixed parabola above determined, and the circle round this triangle, which is identical with the circle OPQ, therefore passes through the focus F of the parabola.

6. The above-mentioned theorem of Newton may also be expressed by saying that *a variable tangent to a conic divides any two fixed tangents to the same homographically*—a theorem which is commonly supposed to be of more recent origin.

SUPPLEMENT TO "EXERCISES."

By *Sir James Cockle, M.A., F.R.S.*

1. To have carried out all that I suggested in commencing the "Exercises" (concluded at p. 119 of vol. VI. N. S.) would have unduly encroached on these pages. But I have followed up and expanded the suggestions elsewhere.* Remarking that there is a continuation of No. I. (at p. 65 of vol. I, N. S.). I shall now make a brief addition to the matter appearing in this Journal.

2. In illustration of the theorem† in Art. 4 of No. V. (N. S., v. 151) let $m = 2$, then the operator

$$1 - \frac{q^4}{[D]_2^4}\, x^4$$

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* See papers On Fractional Criticoids (*Phil. Mag.* for May, 1871); On Hyperdistributives (*ib.* for April, 1872); On Primary Forms (*ib.* for February, 1875); and On a Differential Criticoid (*ib.* for December, 1875). If to the relations (9) and (10) of Art. 8 of Exercise No. I we superadd (with the two new symbols u_3 and a_3),

$$u_3 + 3a_1u_2 + 3a_2u_1 + a_3 = A_3,$$

we shall have identically

$$F_3(u) + F_3(a) = F_3(A),$$

provided that

$$F_3(u) = u_3 - 3u_1u_2 + 2u_1^3.$$

When $n = 2$, or 3, then $F_n(u)$ is deduced from A_n, on therein replacing a_r by $(-u_1)^r$. When $n > 3$ then A_n, meaning thereby an expression which analogies of notation enable us to write as $(u + a)_n$, ceases to yield functions possessing the hyperdistributive property.

† The theorems in Art. 3 and 4 of No. V. were suggested by my noticing that if, in (2) of No. IV. (vol. IV. p. 150), we suppose that the integer $n = -2a$, then (2) and (3) together give

$$x^{2a+1}\phi(a+1) = \phi(-a) = \left(\frac{1}{x}\frac{d}{dx}\right)^{-2a}\phi(a) = \left(\frac{1}{x}\frac{d}{dx}\right)^{-2a-1}\phi(a+1),$$

being identically

$$(D-2)\left(1\pm\frac{q^2}{[D]_2^2}x^2\right)\left(1\mp\frac{q^2}{[D]_2^2}x^2\right)\frac{1}{D-2},$$

we see that the quartordinal breaks up into biordinals.

3. The equation (ix) of Art. 4 of No. VI. (*ib.* VI. 117) is rationalized by putting

$$-4\beta^2=(x-x^{-1})^2;\ \text{or, say, } 2i\beta^2=x-x^{-1},$$

and changing the independent variable from β to x. We thus transform (ix) into

$$\frac{d^2u}{dx^2}+2P_2\frac{du}{dx}+Q_2u=0,$$

wherein

$$2P_2=\tfrac{2}{3}\left\{2x^2-1+\frac{4x^2}{iM(x^2-1)-2(x^2+1)}\right\}\frac{1}{x(x^2-1)},$$

$$Q_2=-\tfrac{2}{3}\frac{1}{x^2}.$$

4. This biordinal is soluble for all constant values whatever of M. When $M=\infty$, and when $iM=2$ or -2, it becomes a binomial.

5. Taking $M=\infty$, we have, on changing signs,

$$(1-x^2)x^2\frac{d^2u}{dx^2}+(\tfrac{2}{3}-\tfrac{4}{3}x^2)x\frac{du}{dx}-\tfrac{2}{3}(1-x^2)u=0,$$

or
$$(D+\tfrac{1}{3})(D-\tfrac{2}{3})u-(D-\tfrac{4}{3})(D-\tfrac{1}{3})x^2u=0,$$

which, like the transformation of (xvi.), noticed in No. VI. (Art. 10) breaks up into factors.

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whence, if we put $x^{2a+1}\phi(a+1)=\chi$, we get

$$\left(\frac{1}{x}\frac{d}{dx}\right)^{2a+1}\chi=\frac{1}{x^{2a+1}}\chi.$$

The general resemblance of the forms of ϕ is of cotirse preserved when n is negative. Thtis, taking a particular case of (30) of No. I. and writing x instead of ξ, we have

$$\int e^{\sqrt{(2x)}}dx=\int e^{\sqrt{(2x)}}\frac{\sqrt{(2x)}}{\sqrt{(2x)}}dx=e^{\sqrt{(2x)}}\sqrt{\{(2x)-1\}}.$$

6. Take $iM=2$, and we obtain

$$x^2\frac{d^2u}{dx^2}+\tfrac{2}{3}x\frac{du}{dx}-\tfrac{2}{9}u=0,$$

or
$$(D^2-\tfrac{1}{3}D-\tfrac{2}{9})\,u=(D+\tfrac{1}{3})\,(D-\tfrac{2}{3})\,u=0.$$

7. Take $iM=-2$ and we obtain

$$x^2\frac{d^2u}{dx^2}+\tfrac{4}{3}x\frac{du}{dx}-\tfrac{2}{9}u=0,$$

or
$$(D^2+\tfrac{1}{3}D-\tfrac{2}{9})\,u=(D+\tfrac{2}{3})\,(D-\tfrac{1}{3})\,u=0.$$

8. As well to verify as to harmonize our results, put

$$b=-C\{\tfrac{1}{2}(t-t^{-1})\}^{\frac{2}{3}}.$$

Then (iv) reduces to

$$\frac{db}{dx}=(2C^2)^{\frac{1}{3}}\{\tfrac{1}{2}(t+t^{-1})\};$$

and differentiation gives

$$\frac{db}{dt}=-\tfrac{2}{3}C\{\tfrac{1}{2}(t-t^{-1})\}^{-\frac{1}{3}}(1+t^{-2}).$$

Consequently

$$\frac{dx}{dt}=-\tfrac{1}{3}\left(\frac{2}{C}\right)^{\frac{1}{3}}\frac{1}{t}\{\tfrac{1}{2}(t-t^{-1})\}^{-\frac{1}{3}},$$

whence, by logarithmic differentiation, and reduction,

$$\frac{dt}{dx}\frac{d^2x}{dt^2}=-\tfrac{1}{3}\frac{4t^2-2}{t(t^2-1)}.$$

9. Again, since

$$\frac{dx}{dt}=-\tfrac{1}{3}(2^{\frac{1}{2}}i)\frac{1}{ib^{\frac{1}{2}}},$$

therefore
$$\frac{1}{a}\frac{dt}{dx}=\left(Kb^{\frac{1}{2}}-\frac{2b}{C^3}\frac{db}{dx}\right)\left(-\frac{3}{2^{\frac{1}{2}}}ib^{\frac{1}{2}}\right)$$

$$=3t\left(-\frac{iK}{2^{\frac{1}{2}}}b^3+\frac{2^{\frac{1}{2}}}{C^3}ib^{\frac{1}{2}}\frac{db}{dx}\right)$$

$$=3t\left\{\frac{iKC^3}{4.2^{\frac{1}{2}}}(t-t^{-1})^2-\frac{2^{\frac{1}{2}}}{2C^{\frac{1}{2}}}(t-t^{-1})\frac{db}{dx}\right\}$$

$$=3t\left\{\frac{iKC^3}{2^{\frac{1}{2}}4}(t-t^{-1})^2-\tfrac{2}{4}(t^2-t^{-2})\right\}.$$

10. It follows that if in (5) we change the independent variable, from x to t, we shall thereby transform it into

$$\frac{d^2u}{dt^2} + 2R\frac{du}{dt} + Su = 0,$$

wherein

$$2R = \tfrac{2}{3}\left\{2t^2 - 1 + \frac{4t^2}{2^{-\frac{1}{3}}iKC^3(t^2-1) - 2(t^2+1)}\right\}\frac{1}{t(t^2-1)},$$

$$S = b\left(\frac{dx}{dt}\right)^2 = -\tfrac{2}{3}\frac{1}{t^2}.$$

11. This verifies the preceding result; for P_2, $Q_2 = R$, S, if in the latter we replace t by x. In Art. 12 of No. VI. the symbol H should be replaced by $2H$.

2, Sandringham Gardens,
Ealing, near London, W.,
April 19*th*, 1880.

ERRATA IN NEW SERIES.

Vol. I. p. 8, to the dexter of equation (*B*) prefix $x^2 +$; *ib.* p. 10, line 3, the third "=" should be a comma; *ib.* p. 12, dele the comma in line 4 of Art. 12; Vol. VI. p. 119, Art. 12, replace H by $2H$ and "(xxvii)" by "(xxvi)."

GENERAL EXPRESSION FOR THE RADIUS OF CURVATURE IN DIPOLAR COORDINATES.

By *Percival Frost, M.A.*

I AM not aware that an expression has been obtained for the radius of curvature in terms of the partial derivatives of $f(r, r')$, when $f(r, r') = 0$ is the equation of a curve.

The expression which I have investigated has no pretensions to beauty, but as it gives a correct result in the case of the lemniscate and ellipse I suppose that it is accurate.

It will be convenient to use the following notation

$$du = Pdr + P'dr' = 0,$$

$$dP = qdr + q'dr'$$

$$dP' = q'dr + q''dr'.$$

Let ϕ, ϕ' (fig. 1) be the angles made by the tangents at P with SP, $S'P$, so that

$$\cos\phi = \frac{dr}{ds}, \quad \cos\phi' = -\frac{dr'}{ds}.$$

Let T be any point in the tangent at P, TM, TM' perpendicular to SP, SP', then PT, PM, PM' are finite quantities proportional to the ultimate values of ds, dr, $-dr'$.

Let MN be a diameter of the circle on PT as diameter, so that $\angle MNM' = \phi + \phi' = \psi$; therefore

$$(ds)^2 \sin^2\psi = (dr)^2 + (dr')^2 + 2dr\,dr' \cos\psi,$$

$$\{(ds)^2 - (dr)^2\} \sin^2\psi = (dr \cos\psi + dr')^2;$$

therefore

$$\sin^2\phi = \frac{(ds)^2 - (dr)^2}{ds^2} = \frac{(P' \cos\psi - P)^2}{P^2 + P'^2 - 2PP' \cos\psi} = \frac{V^2}{Q^2} \text{ suppose.}$$

Since $rdr = \rho dp$, $\dfrac{rdr}{\rho} = d(r \sin\phi) = dr \sin\phi + rd \sin\phi$,

$$\text{let } \frac{dr}{P'} = \frac{dr'}{-P} = \sigma,$$

$$\frac{\sigma P'}{\rho} = \frac{\sigma P'}{r} \cdot \frac{V}{Q} + \frac{dV}{Q} - \frac{VdQ}{Q^2};$$

similarly $$-\frac{\sigma P}{\rho} = -\frac{\sigma P}{r'} \cdot \frac{V'}{Q} + \frac{dV'}{Q} - \frac{V'dQ}{Q^2},$$

where $$V = P - P' \cos\psi, \quad V' = P' - P \cos\psi,$$

and, therefore, $$Q^2 = PV + P'V';$$

hence $$\frac{\sigma Q^2}{\rho} = \sigma \left(\frac{P}{r'} + \frac{P'}{r}\right) \frac{VV'}{Q} + \frac{V'dV - VdV'}{Q},$$

$$V'dV - VdV' = V' (dP - dP' \cos\psi - P'd \cos\psi)$$
$$- V (dP' - dP \cos\psi - Pd \cos\psi)$$
$$= (P'dP - PdP') \sin^2\psi + (P^2 - P'^2)\, d \cos\psi.$$

Now $$dP = \sigma (P'q - Pq'), \quad -dP' = \sigma (P'q' - Pq'');$$

therefore $$P'dP - PdP' = \sigma (P'^2 q - 2PP'q' + P^2 q'');$$

also $$c^2 = r^2 + r'^2 + 2rr' \cos\psi;$$

therefore

$$0 = \sigma \{P' (r + r' \cos\psi) - P(r' + r \cos\psi)\} - rr'd \cos\psi,$$

$$rr'd \cos\psi = -\sigma (rV' - r'V);$$

therefore

$$\frac{Q^3}{\rho} - (P'^2q - 2PP'q' + P^2q'')\sin^2\psi$$

$$= \frac{1}{rr'}\{(Pr + P'r')\,VV' - (P^2 - P'^2)\,(V'r - V'r)\}$$

$$= \frac{1}{rr'}[V'r\{P(P - P'\cos\psi) - P^2 + P'^2\}]$$

$$+ Vr'\{P^2 - P'^2 + P'(P' - P\cos\psi)\}$$

$$= V^2\frac{P}{r} + V'^2\frac{P'}{r'} = Q^2\left(\frac{P}{r} + \frac{P'}{r'}\right) - \left(\frac{P'}{r} + \frac{P}{r'}\right)PP'\sin^2\psi.$$

Since $V^2 = Q^2 - P'^2\sin^2\psi$, and $V'^2 = Q^2 - P^2\sin^2\psi$,

whence ρ is determined.

In the lemniscate $rr' = a^2$, $c = 2a$,

$$P = r',\ P' = r,\ q = q'' = 0,\ q' = 1,$$

$$4a^2 = r^2 + r'^2 + 2a^2\cos\psi,$$

$$Q^2 = r^2 + r'^2 - 2rr'\cos\psi = 2(r^2 + r'^2) - 4a^2 = 2(r - r')^2,$$

$$\frac{Q^3}{\rho} = -2rr'\sin^2\psi + Q^2\frac{r'^2 + r^2}{rr'} - 2rr'\sin^2\psi$$

$$= -4a^2 + \frac{\{2a^2 - (r - r')^2\}^2}{a^2} + 2(r - r')^2\frac{(r - r')^2 + 2a^2}{a^2}$$

$$= \frac{3(r - r')^4}{a^2};$$

therefore $$\rho = \frac{2\sqrt{2a^2}}{3(r \sim r')}.$$

In the case of $r + r' = 2a$, $c = 2ae$,

$$P = 1 = P',\ q = q' = q'' = 0,$$

$$Q^2 = 2 - 2\cos\psi,$$

$$4a^2e^2 = r^2 + r'^2 + 2rr'\cos\psi$$

$$= 4a^2 - 2rr'(1 - \cos\psi);$$

therefore $$Q^2 = \frac{4b^2}{CD^2},$$

$$\frac{Q^3}{\rho} = Q^3\left(\frac{1}{r} + \frac{1}{r'}\right) - \left(\frac{1}{r} + \frac{1}{r'}\right)\sin^2\psi;$$

therefore $$\frac{Q}{\rho} = \frac{2a}{rr'} - \frac{a}{rr'}(1+\cos\psi)$$

$$= \frac{Q^2 a}{2CD^2};$$

therefore $$\rho = \frac{CD^3}{ab} = \frac{CD^2}{PF}.$$

ON THE CURVES REPRESENTED BY THE EQUATION

$$\frac{dx}{\sqrt{\{(1-x^2)(1-k^2x^2)\}}} \pm \frac{dy}{\sqrt{\{(1-y^2)(1-k^2y^2)\}}} = 0$$

AND THEIR ENVELOPES.

By *H. H. Turner*, Scholar of Trinity College, Cambridge.

1. In a paper on "Singular Solutions," in a previous number (*Messenger*, vol. VI. p. 23, June, 1876), Prof. Cayley has discussed the curves represented by the equation

$$\frac{dx}{\sqrt{(1-x^2)}} \pm \frac{dy}{\sqrt{(1-y^2)}} = 0.$$

The integral of this equation is

$$\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}c,$$

which reduces to

$$x^2 + y^2 + c^2 + 2cxy = 1,$$

and represents a system of conics. From either equation we see that the lines $x^2=1$ and $y^2=1$ are envelopes of the system, which is a series of ellipses inscribed in the square formed by these lines, together with hyperbolas in the corners formed by producing adjacent sides of the square; the axes of all the curves being the lines

$$x \pm y = 0.$$

2. The following is a discussion of the curves represented by the fundamental equation in elliptic functions

$$\frac{dx}{\sqrt{\{(1-x^2)(1-k^2x^2)\}}} \pm \frac{dy}{\sqrt{\{(1-y^2)(1-k^2y^2)\}}} = 0 \ldots (1).$$

Putting $x = \sin\phi$ and $y = \sin\psi$, the integral is known to be

$$\cos\mu = \cos\phi\cos\psi \pm \sin\phi\sin\psi\sqrt{(1 - k^2\sin^2\mu)},$$

and substituting back for ϕ and ψ, and writing c for $\sin\mu$, we have

$$\sqrt{(1-c^2)} = \sqrt{\{(1-x^2)(1-y^2)\}} \pm xy\sqrt{(1-k^2c^2)};$$

therefore

$$(1-x^2)(1-y^2) = x^2y^2(1-k^2c^2) + 1 - c^2 \pm 2xy\sqrt{\{(1-c^2)(1-k^2c^2)\}},$$

or $$x^2 + y^2 - c^2(1 + x^2y^2k^2) = \pm 2xy\sqrt{\{(1-c^2)(1-k^2c^2)\}} \ldots (2).$$

This equation represents curves of the fourth degree, but they only differ from conics in the terms $k^2c^2x^2y^2$, which vanishes when k or c is zero. There are four systems of envelopes in this case, as may be seen from either the integral or differential equation, viz.

$$x^2 = 1,\ y^2 = 1,\ k^2x^2 = 1,\ k^2y^2 = 1.$$

The curves are still symmetrical with respect to the lines $x \pm y = 0$, to which they reduce when $c = 0$.

We need only consider one sign of the radical, as the effect of a change of sign is merely to turn the curves through a right angle.

3. The most convenient form of the equation is

$$x = \frac{\pm y\sqrt{\{(1-c^2)(1-k^2c^2)\}} \pm c\sqrt{\{(1-y^2)(1-k^2y^2)\}}}{1 - k^2c^2y^2} \ldots (3).$$

We might of course express y similarly in terms of x. We shall take the upper sign to the first radical.

Representing the numerator and denominator of this fraction by N and D respectively, the following identities are useful:

$$D^2 - N^2 = [\sqrt{\{(1-c^2)(1-y^2)\}} \pm cy\sqrt{\{(1-k^2c^2)(1-k^2y^2)\}}]^2 \ldots (4),$$

$$D^2 - k^2N^2 = [\sqrt{\{(1-k^2c^2)(1-k^2y^2)\}} \pm k^2cy\sqrt{\{(1-c^2)(1-y^2)\}}]^2 \ldots (5),$$

$$[y\sqrt{\{(1-c^2)(1-k^2c^2)\}}]^2 - [c\sqrt{\{(1-y^2)(1-k^2y^2)\}}]^2 = (y^2-c^2)(1-k^2y^2c^2) \ldots (6).$$

4. The system naturally divides into four classes.

I. Let $c^2 < 0$, and write $-c^2$ for it. Then equation (3) becomes

$$x = \frac{y\sqrt{\{(1+c^2)(1+k^2c^2)\}} \pm c\sqrt{\{(y^2-1)(1-k^2y^2)\}}}{1 + k^2c^2y^2}.$$

Thus $y^2 > 1$ and $< \frac{1}{k^2}$. Similarly $x^2 > 1$ and $< \frac{1}{k^2}$, and since

$$[y\sqrt{\{(1+c^2)(1+k^2c^2)\}}]^2 - [c\sqrt{\{(y^2-1)(1-k^2y^2)\}}]^2 = (y^2+c^2)(1+k^2y^2c^2),$$

which is positive, the sign of x is the same as that of y.

Since two values of x correspond to any given value of y, and the value of both are continuous between the above limits, we get two oval curves in the $(++)$ and $(--)$ compartments, enclosed by the lines

$$y^2 = 1, \quad y^2 = \frac{1}{k^2},$$

$$x^2 = 1, \quad x^2 = \frac{1}{k^2}.$$

If we had taken

$$x = \frac{-y\sqrt{\{(1+c^2)(1+k^2c^2)\}} \pm c\sqrt{\{(y^2-1)(1-k^2y^2)\}}}{1+k^2y^2c^2},$$

we should have got similar curves in the $(+-)$ and $(-+)$ compartments. The curves all touch the above lines, which are thus envelopes of the system. A set of these curves is drawn in fig. 2.

When $k = 0$ the outer lines go off to infinity, and these curves become the hyperbolas of the conic system

$$\frac{dx}{\sqrt{(1-x^2)}} \pm \frac{dy}{\sqrt{(1-y^2)}} = 0.$$

II. Let c^2 lie between 0 and 1. Then

$$x = \frac{y\sqrt{\{(1-c^2)(1-k^2c^2)\}} \pm c\sqrt{\{(1-y^2)(1-k^2y^2)\}}}{1-k^2c^2y^2}.$$

Thus y^2 is either < 1 or $> \frac{1}{k^2}$.

(a) When $y^2 < 1$, from (4)

$$D^2 - N^2 > 0,$$

therefore $$x^2 < 1.$$

Thus the curves are contained in the squares $x^2 = 1$, $y^2 = 1$, the sides of which they all touch, and are a series of ovals which reduce to ellipses when $k = 0$.

(*b*) When $y^2 > \frac{1}{k^2}$, we write

$$x = \frac{y\sqrt{\{(1-c^2)(1-k^2c^2)\}} \pm c\sqrt{\{(k^2y^2-1)(y^2-1)\}}}{1-k^2c^2y^2},$$

and from (5) $\qquad k^2N^2 - D^2 > 0;$

therefore $\qquad x^2 > \frac{1}{k^2}.$

Thus the curves lie wholly in the corner compartments of the lines $k^2x^2 = 1$, $k^2y^2 = 1$.

Consider y positive. Call the roots with upper and lower sign α and β respectively. Then as y increases from $\frac{1}{k}$, D decreases, and α increases indefinitely till $y = \frac{1}{kc}$, which line is an asymptote. Then D is negative and α passes through infinity and becomes negative; and as y increases, α decreases, until when y is infinite $\alpha = -\frac{1}{kc}$.

Now considering β, we notice from (6) since $y^2 > c^2$, that N and D are both positive when $y^2 < \frac{1}{k^2c^2}$, and negative when $y^2 > \frac{1}{k^2c^2}$. Thus β is always positive. It assumes an indeterminate form when $y = \frac{1}{kc}$. Referring to the original equation, we find

$$\beta = \frac{1-k^2c^4}{2\sqrt{\{(1-c^2)(1-k^2c^2)k^2c^2\}}}.$$

Remembering that the curve is symmetrical with respect to the lines $x \pm y = 0$, we now get some idea of its form, which is shewn in fig. 3.

The lines $kcx = \pm 1$ and $kcy = \pm 1$ are asymptotes. The other corners can be filled in similarly by using the equation

$$x = \frac{-y\{(1-c^2)(1-k^2c^2)\} \pm c\sqrt{\{(1-y^2)(1-k^2y^2)\}}}{1-k^2c^2y^2}.$$

The asymptotes vary with each curve. When $k = 0$ the infinite branches go off to infinity and the ovals become the ellipses of the conic system.

II. When c^2 is > 1 and $< \frac{1}{k^2}$ either x or y is imaginary, and the curves are imaginary.

IV. When $c^2 > \frac{1}{k^2}$, we write

$$x = \frac{y\sqrt{\{(c^2-1)(k^2c^2-1)\}} \pm c\sqrt{\{(1-y^2)(1-k^2y^2)\}}}{1-k^2c^2y^2},$$

and, as before, y^2 lies outside the limits 1 and $\frac{1}{k^2}$.

(*a*) Let $y^2 < 1$. Then from (5)

$$k^2N^2 - D^2 > 0;$$

therefore

$$x^2 > \frac{1}{k^2}.$$

Also from (6), since $y^2 > c^2$, the ambiguous term governs the sign of N when $y^2 < \frac{1}{k^2c^2}$, and the other term when $y^2 > \frac{1}{k^2c^2}$, But in this latter case, D is negative. Hence, when y is positive and $> \frac{1}{kc}$, both values of x are negative; as y decreases to $\frac{1}{kc}$, one root becomes infinite and then changes sign, while the other assumes an indeterminate form, and is found from the original equation to be

$$-\frac{k^2c^4-1}{2\sqrt{\{(c^2-1)(k^2c^2-1)k^2c^2\}}}.$$

As y further decreases, this negative root increases numerically, and is infinite when $y = -\frac{1}{kc}$, when it becomes positive.

The other root remains positive and finite. The form of the curve is thus that of the horizontal branches in fig. 3.

(*b*) When $y^2 > \frac{1}{k^2}$, $x^2 < 1$ and we fill in the vertical branches in a similar way.

When $k = 0$ these curves do not appear, being wholly at infinity.

TRIGONOMETRICAL THEOREMS INVOLVING THE PRODUCTS OF FOUR SINES OR FOUR COSINES.

By *J. W. L. Glaisher.*

§1. In the *Proceedings of the Cambridge Philosophical Society*, vol. III. pp. 319–329,* I have referred to the following five theorems in elementary trigonometry. If a, b, c, d be any four quantities, and if a', b', c', d', a'', b'', c'', d'' be eight quantities derived from them by the equations

$$a' = \tfrac{1}{2}(-a+b+c+d), \quad a'' = \tfrac{1}{2}(a+b+c+d),$$
$$b' = \tfrac{1}{2}(\ a-b+c+d), \quad b'' = \tfrac{1}{2}(a+b-c-d),$$
$$c' = \tfrac{1}{2}(\ a+b-c+d), \quad c'' = \tfrac{1}{2}(a-b+c-d),$$
$$d' = \tfrac{1}{2}(\ a+b+c-d), \quad d'' = \tfrac{1}{2}(a-b-c+d),$$

then

$$\sin a \sin b \sin c \sin d + \cos a \cos b \cos c \cos d$$
$$= \sin a' \sin b' \sin c' \sin d' + \cos a' \cos b' \cos c' \cos d' \ldots (A),$$

$$\sin a \sin b \sin c \sin d = \sin a' \sin b' \sin c' \sin d'$$
$$+ \sin a'' \sin b'' \sin c'' \sin d'' \ldots (B),$$

$$\cos a \cos b \cos c \cos d = \cos a' \cos b' \cos c' \cos d'$$
$$- \sin a'' \sin b'' \sin c'' \sin d'' \ldots (C),$$

$$\sin^2 a + \sin^2 b + \sin^2 c + \sin^2 d - \sin^2 a' - \sin^2 b' - \sin^2 c' - \sin^2 d'$$
$$= -4 \sin a'' \sin b'' \sin c'' \sin d'' \ldots (D),$$

$$\cos a \cos b \cos c \cos d - \sin a \sin b \sin c \sin d$$
$$= 1 - \tfrac{1}{2}(\sin^2 a' + \sin^2 b' + \sin^2 c' + \sin^2 d') \ldots (E).$$

These equations may be conveniently written, for brevity,

$$\Pi(\sin a) + \Pi(\cos a) = \Pi(\sin a') + \Pi(\cos a') \ldots . (A),$$
$$\Pi(\sin a) = \Pi(\sin a') + \Pi(\sin a'') \ldots \ldots \ldots (B),$$
$$\Pi(\cos a) = \Pi(\cos a') - \Pi(\sin a'') \ldots \ldots \ldots (C),$$
$$\Sigma \sin^2 a - \Sigma \sin^2 a' = -4\Pi(\sin a'') \ldots \ldots \ldots (D),$$
$$\Pi(\cos a) - \Pi(\sin a) = 1 - \tfrac{1}{2}\Sigma \sin^2 a' \ldots \ldots \ldots (E).$$

* *Theorems in elementary trigonometry*, vol. III. part VIII. (February 9, 1880).

The first result (A) is derivable, by addition, from (B) and (C).

The theorems $(A), \ldots (E)$ were originally deduced from elliptic-function formulæ given in the paper referred to, and they are also there proved in an elementary manner by putting

$$a = \alpha + \beta,$$
$$b = \alpha - \beta,$$
$$c = \gamma + \delta,$$
$$d = \gamma - \delta,$$

from which it follows that

$$a' = \gamma - \beta, \quad a'' = \alpha + \gamma,$$
$$b' = \gamma + \beta, \quad b'' = \alpha - \gamma,$$
$$c' = \alpha - \delta, \quad c'' = \beta + \delta,$$
$$d' = \alpha + \delta, \quad d'' = \beta - \delta.$$

When the arguments have this form the theorems at once reduce, in virtue of the formulæ

$$\sin(p+q)\sin(p-q) = \sin^2 p - \sin^2 q,$$
$$\cos(p+q)\cos(p-q) = 1 - \sin^2 p - \sin^2 q,$$

to algebraical identities, which may be readily verified.

In the present paper I propose to examine the results $(A), \ldots (E)$ more in detail, and to consider the general class of theorems to which they belong.

§ 2. In the first place it is to be remarked that there is a certain reciprocity existing between the quantities $a, \ldots d$, $a', \ldots d'$, $a'', \ldots d''$, for, from the definitions of the singly and doubly accented letters in § 1, it follows that

$$a' = \tfrac{1}{2}(a'' - b'' - c'' - d''), \quad a'' = \tfrac{1}{2}(\ a' + b' + c' + d'),$$
$$b' = \tfrac{1}{2}(a'' - b'' + c'' + d''), \quad b'' = \tfrac{1}{2}(-a' - b' + c' + d'),$$
$$c' = \tfrac{1}{2}(a'' + b'' - c'' + d''), \quad c'' = \tfrac{1}{2}(-a' + b' - c' + d'),$$
$$d' = \tfrac{1}{2}(a'' + b'' + c'' - d''), \quad d'' = \tfrac{1}{2}(-a' + b' + c' - d'),$$

so that if $(a')'$, $(b')'$, $(c')'$, $(d')'$ denote the quantities formed from a', b', c', d' by the same process as a', b', c', d' were formed from a, b, c, d, and if a corresponding meaning be assigned in

the case of the double accent, we have the four sets of equations,

$$\begin{aligned}
&(a')' = a, \quad (a'')'' = a, \quad (a'')' = -a', \quad (a')'' = a'',\\
&(b')' = b, \quad (b'')'' = b, \quad (b'')' = b', \quad (b')'' = -b'',\\
&(c')' = c, \quad (c'')'' = c, \quad (c'')' = c', \quad (c')'' = -c'',\\
&(d')' = d, \quad (d'')'' = d, \quad (d'')' = d', \quad (d')'' = -d'',
\end{aligned}$$

§ 3. Now

$$\begin{aligned}
8\cos a\cos b\cos c\cos d = {}&\cos(a+b+c+d) + \cos(\ a+b+c-d)\\
&+\cos(a+b-c+d) + \cos(a-b+c+d) + \cos(-a+b+c+d)\\
&+\cos(a+b-c-d) + \cos(a-b+c-d) + \cos(\ a-b-c+d),
\end{aligned}$$

and

$8\sin a\sin b\sin c\sin d$ = an expression differing from the above only in having the signs of the second, ...fifth terms negative.

Thus, with the notation employed in this paper,

$$\begin{aligned}
8\cos a\cos b\cos c\cos d = {}&\cos 2a' + \cos 2b' + \cos 2c' + \cos 2d'\\
&+\cos 2a'' + \cos 2b'' + \cos 2c'' + \cos 2d'',\\
8\sin a\sin b\sin c\sin d = {}&-\cos 2a' - \cos 2b' - \cos 2c' + \cos 2d'\\
&+\cos 2a'' + \cos 2b'' + \cos 2c'' + \cos 2d'',
\end{aligned}$$

which may be concisely written

$$8\Pi(\cos a) = \Sigma\cos 2a' + \Sigma\cos 2a'' \ldots\ldots (1),$$

$$8\Pi(\sin a) = -\Sigma\cos 2a' + \Sigma\cos 2a'' \ldots\ldots (2);$$

from these by addition and subtraction we have

$$4\{\Pi(\cos a) - \Pi(\sin a)\} = \Sigma\cos 2a',$$

$$4\{\Pi(\cos a) + \Pi(\sin a)\} = \Sigma\cos 2a''.$$

The former of these two results is in fact the equation (E).

§ 4. Substituting in (1) and (2) a', b', ... and a'', b'', ... respectively, in place of a, b, ..., and making use of the results of § 2, we have

$$8\Pi(\cos a') = \Sigma\cos 2a + \Sigma\cos 2a'' \ldots\ldots (3),$$

$$8\Pi(\sin a') = -\Sigma\cos 2a + \Sigma\cos 2a'' \ldots\ldots (4),$$

$$8\Pi(\cos a'') = \Sigma\cos 2a' + \Sigma\cos 2a \ldots\ldots (5),$$

$$8\Pi(\sin a'') = -\Sigma\cos 2a' + \Sigma\cos 2a \ldots\ldots (6),$$

and writing for the moment $(2a)$, $(2a')$, ... in place of $\Sigma\cos 2a$, $\Sigma\cos 2a'$, ... the formulæ (1), ...(6) are

$$8\Pi(\cos a) = (2a') + (2a''), \quad 8\Pi(\sin a) = -(2a') + (2a''),$$
$$8\Pi(\cos a') = (2a'') + (2a), \quad 8\Pi(\sin a') = (2a'') - (2a),$$
$$8\Pi(\cos a'') = (2a) + (2a'), \quad 8\Pi(\sin a'') = (2a) - (2a'),$$

and from these we deduce at once that

$$\Pi(\sin a) = \Pi(\sin a') + \Pi(\sin a''),$$
$$\Pi(\cos a) = \Pi(\cos a') - \Pi(\sin a''),$$

which are the theorems (B) and (C).

The theorem (6) is in fact (D), and as has been already remarked (A) is deducible from (B) and (C) by addition. We have thus obtained the five theorems (A), ...(E).

§ 5. It may be observed that we also have

$$4\{\Pi(\cos a)+\Pi(\cos a')+\Pi(\cos a'')\} = \Sigma\cos 2a + \Sigma\cos 2a' + \Sigma\cos 2a'',$$
$$4\{-\Pi(\cos a)+\Pi(\cos a')+\Pi(\cos a'')\} = \Sigma\cos 2a,$$
$$4\{\Pi(\sin a)+\Pi(\sin a')+\Pi(\sin a'')\} = \Sigma\cos 2a'' - \Sigma\cos 2a',$$
$$\&c. = \&c.$$

§ 6. The equation (C) is derivable from (B) by substituting $\frac{1}{2}\pi - a$, $\frac{1}{2}\pi - b$, ... for a, b, ... and (A) being a direct consequence of (B) and (C) it follows that the three more important formulæ (A), (B), (C) of the group (A), ...(E) are in fact involved in (B).

As particular cases of (B) and (C) we may put α, β, γ, δ $= 1, 2, 3, 4$ and $= 1, 3, 5, 7$ respectively, α, ...δ being as in § 1, and we thus obtain the identical equations

$$\sin^2 x \sin 3x \sin 7x + \sin^2 5x \sin x \sin 3x - \sin^2 2x \sin 4x \sin 6x = 0,$$
$$\cos^2 x \cos 3x \cos 7x - \cos^2 5x \cos x \cos 3x + \sin^2 2x \sin 4x \sin 6x = 0,$$

and

$$\sin^2 x \sin 2x \sin 6x + \sin^2 4x \sin x \sin 3x - \sin^2 2x \sin 3x \sin 5x = 0,$$
$$\cos^2 x \cos 2x \cos 6x - \cos^2 4x \cos x \cos 3x + \sin^2 2x \sin 3x \sin 5x = 0.$$

§ 7. If we put $d = a + b + c$, then

$$a' = b + c, \quad a'' = a + b + c,$$
$$b' = c + a, \quad b'' = -c,$$
$$c' = a + b, \quad c'' = -b,$$
$$d' = 0, \quad d'' = a,$$

and the formulæ at the end of § 3 become

$$\cos a \cos b \cos c \cos(a+b+c) - \sin a \sin b \sin c \sin(a+b+c)$$
$$= \tfrac{1}{4}\{1 + \cos 2(b+c) + \cos 2(c+a) + \cos 2(a+b)\} \ldots (7),$$
$$\cos a \cos b \cos c \cos(a+b+c) + \sin a \sin b \sin c \sin(a+b+c)$$
$$= \tfrac{1}{4}\{\cos 2a + \cos 2b + \cos 2c + \cos 2(a+b+c)\} \ldots\ldots (8),$$

while (C) becomes

$$\cos a \cos b \cos c \cos(a+b+c) + \sin a \sin b \sin c \sin(a+b+c)$$
$$= \cos(b+c)\cos(c+a)\cos(a+b) \ldots\ldots\ldots\ldots (9),$$

which is readily identified with (8).

If, in (7) and (9), $a = b = c = x$, they become

$$\cos^3 x \cos 3x - \sin^3 x \sin 3x = \tfrac{1}{4}(1 + 3\cos 2x),$$
$$\cos^3 x \cos 3x + \sin^3 x \sin 3x = \cos^3 2x,$$

the latter of which is well-known formula.

The equation (9) may also be written

$$\cos a \cos b \cos c = \cos s \cos(s-a)\cos(s-b)\cos(s+c)$$
$$+ \sin s \sin(s-a)\sin(s-b)\sin(s-c),$$

where $s = \tfrac{1}{2}(a+b+c)$,

and in this form it is involved in the well-known formulæ for the spherical excess

$$\sin^2 \tfrac{1}{4}E = \frac{\sin \frac{1}{2}s \sin \frac{1}{2}(s-a) \sin \frac{1}{2}(s-b) \sin \frac{1}{2}(s-c)}{\cos \frac{1}{2}a \cos \frac{1}{2}b \cos \frac{1}{2}c},$$

$$\cos^2 \tfrac{1}{4}E = \frac{\cos \frac{1}{2}s \cos \frac{1}{2}(s-a) \cos \frac{1}{2}(s-b) \cos \frac{1}{2}(s-c)}{\cos \frac{1}{2}a \cos \frac{1}{2}b \cos \frac{1}{2}c}.$$

Adding and subtracting (7) and (9) we see that

$$8 \cos s \cos(s-a) \cos(s-b) \cos(s-c)$$
$$= 1 + \cos 2a + \cos 2b + \cos 2c + 4 \cos a \cos b \cos c;$$
$$8 \sin s \sin(s-a) \sin(s-b) \sin(s-c)$$
$$= -1 - \cos 2a - \cos 2b - \cos 2c + 4 \cos a \cos b \cos c,$$

the second of these is a formula used in spherical trigonometry*.

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* Todhunter's *Spherical Trigonometry*, second edition (1863), p. 71.

§ 8. The theorem (B) when the arguments a, b, ... are replaced by $\alpha+\beta$, $\alpha-\beta$, ... as in § 1, becomes

$$\sin(a+\beta)\sin(\alpha-\beta)\sin(\gamma+\delta)\sin(\gamma-\delta)$$
$$+\sin(\beta+\gamma)\sin(\beta-\gamma)\sin(\alpha+\delta)\sin(\alpha-\delta)$$
$$+\sin(\gamma+\alpha)\sin(\gamma-\alpha)\sin(\beta+\delta)\sin(\beta-\delta)=0,$$

i.e., putting x, y, z, w, for $\sin\alpha$, $\sin\beta$, $\sin\gamma$, $\sin\delta$ respectively, this is

$$(x^2-y^2)(z^2-w^2)+(y^2-z^2)(x^2-w^2)+(z^2-x^2)(y^2-w^2)=0,$$

which is a well known identity.

It thus appears that (B) is in fact a simple deduction from the identity

$$(\beta-\gamma)(\alpha-\delta)+(\gamma-\alpha)(\beta-\delta)+(\alpha-\beta)(\gamma-\delta)=0,$$

and that every algebraical identity involving only *differences* of letters $\beta-\gamma$, $\gamma-\alpha$, &c., gives rise to a trigonometrical identity involving sines, by replacing $\beta-\gamma$ by $\sin^2\beta-\sin^2\gamma$, that is by $\sin(\beta+\gamma)\sin(\beta-\gamma)$, $\gamma-\alpha$ by $\sin(\gamma+\alpha)\sin(\gamma-\alpha)$, &c.

§ 9. If we put

$$a=\beta-\gamma,\quad f=\alpha-\delta,$$
$$b=\gamma-\alpha,\quad g=\beta-\delta,$$
$$c=\alpha-\beta,\quad h=\gamma-\delta,$$

then in any identity involving a, b, c, f, g, h, we may replace

a by $\sin(\beta+\gamma)\sin(\beta-\gamma)$, $=-\sin a'\sin b'$,
b „ $\sin(\gamma+\alpha)\sin(\gamma-\alpha)$, $=-\sin a''\sin b''$,
c „ $\sin(\alpha+\beta)\sin(\alpha-\beta)$, $=\sin a\sin b$,
f „ $\sin(\alpha+\delta)\sin(\alpha-\delta)$, $=\sin c'\sin d'$,
g „ $\sin(\beta+\delta)\sin(\beta-\delta)$, $=\sin c''\sin d''$,
h „ $\sin(\gamma+\delta)\sin(\gamma-\delta)$, $=\sin c\sin d$.

Thus, as we have seen, from the identity

$$af+bg+ch=0^*,$$

we obtain

$$\sin a'\sin b'\sin c'\sin d'+\sin a''\sin b''\sin c''\sin d''$$
$$-\sin a\sin b\sin c\sin d=0,$$

* Note that in the algebraical formulæ the letters a, b, c stand for $\beta-\gamma$, $\gamma-\alpha$, $\alpha-\beta$; in the trigonometrical formulæ a, b, c, d are any four quantities, and a', ...d', a'', ...d'' eight quantities derived from them as in § 1.

which is (B); from the identity

$$agh + bhf + cfg + abc = 0,$$

we obtain

$$\begin{aligned}&\sin a' \sin b' \sin c \sin d \sin c'' \sin d''\\ &\qquad + \sin a'' \sin b'' \sin c \sin d \sin c' \sin d'\\ &- \sin a \sin b \sin c' \sin d' \sin c'' \sin d''\\ &\qquad - \sin a \sin b \sin a' \sin b' \sin a'' \sin b'' = 0,\end{aligned}$$

which may be written

$$\begin{aligned}&\sin a \sin b (\sin a' \sin b' \sin a'' \sin b'' + \sin c' \sin d' \sin c'' \sin d'')\\ &= \sin c \sin d (\sin a' \sin b' \sin c'' \sin d'' + \sin a'' \sin b'' \sin c' \sin d').\end{aligned}$$

§ 10. A number of identities involving a, b, c, f, g, h are given in the *Messenger*, vol. VIII. pp. 46, 47 and 54–56, and each of these gives rise to a corresponding trigonometrical theorem; thus, for example, from

$$\begin{aligned}&agh(g^2 + h^2 + a^2)^2 + bhf(h^2 + f^2 + b^2)^2\\ &\qquad + cfg(f^2 + g^2 + c^2)^2 + abc(a^2 + b^2 + c^2)^2 = 0,\end{aligned}$$

(vol. VIII. p. 46), we obtain the equation

$$\begin{aligned}&\sin a' \sin b' \sin c \sin d \sin c'' \sin d''\\ &\qquad \times (\sin^2 a' \sin^2 b' + \sin^2 c \sin^2 d + \sin^2 c'' \sin^2 d'')^2\\ &+ \sin a'' \sin b'' \sin c \sin d \sin c' \sin d'\\ &\qquad \times (\sin^2 a'' \sin^2 b'' + \sin^2 c \sin^2 d + \sin^2 c' \sin^2 d')^2\\ &- \sin a \sin b \sin c' \sin d' \sin c'' \sin d''\\ &\qquad \times (\sin^2 a \sin^2 b + \sin^2 c' \sin^2 d' + \sin^2 c'' \sin^2 d'')^2\\ &= \sin a \sin b \sin a' \sin b' \sin a'' \sin b''\\ &\qquad \times (\sin^2 a \sin^2 b + \sin^2 a' \sin^2 b' + \sin^2 a'' \sin^2 b''),\end{aligned}$$

and the other similar algebraical identities give rise to corresponding formulæ in trigonometry.

§ 11. From the identity

$$a(b-c)^3 + b(c-a)^3 + c(a-b)^3 = 0,$$

(vol. VIII. p. 56), we see that

$$\begin{aligned}&\sin a \sin b (\sin a' \sin b' - \sin a'' \sin b'')^3\\ &- \sin a' \sin b' (\sin a'' \sin b'' + \sin a \sin b)^3\\ &+ \sin a'' \sin b'' (\sin a' \sin b' + \sin a \sin b)^3 = 0,\end{aligned}$$

from

$$\left\{\frac{b-c}{a}+\frac{c-a}{b}+\frac{a-b}{c}\right\}\left\{\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}\right\}=9,$$*

we have

$$\left(\frac{\sin a''\sin b''+\sin a\sin b}{\sin a'\sin b'}-\frac{\sin a\sin b+\sin a'\sin b'}{\sin a''\sin b''}-\frac{\sin a'\sin b'-\sin a''\sin b''}{\sin a\sin b}\right)$$
$$\times\left(\frac{\sin a'\sin b'}{\sin a''\sin b''+\sin a\sin b}-\frac{\sin a''\sin b''}{\sin a\sin b+\sin a'\sin b'}-\frac{\sin a\sin b}{\sin a'\sin b'-\sin a''\sin b''}\right)=9,$$

and from

$$a^3+b^3+c^3-3abc=0,$$

we have

$$\sin^3 a\sin^3 b=\sin^3 a'\sin^3 b'+\sin^3 a''\sin^3 b''$$
$$+3\sin a\sin a'\sin a''\sin b\sin b'\sin b''=0.$$

§12. It appears from what has gone before that whenever we have a sum of products of sines of *differences* of angles equated to zero, say

$$\Sigma.\Pi\{\sin(\alpha-\beta)\}=0,$$

the number of sines forming the product being the same in each term, then we infer that

$$\Sigma.\Pi\{\sin(\alpha-\beta)\sin(\alpha+\beta)\}=0.$$

For example, since

$$\sin(\beta-\gamma)\sin(\alpha-\delta)+\sin(\gamma-\alpha)\sin(\beta-\delta)+\sin(\alpha-\beta)\sin(\gamma-\delta)=0\ \ldots\ldots\ldots\ldots(a),$$

we infer that

$$\sin(\beta-\gamma)\sin(\alpha-\delta)\sin(\beta+\gamma)\sin(\alpha+\delta)+\sin(\gamma-\alpha)\sin(\beta-\delta)\sin(\gamma+\alpha)\sin(\beta+\delta)+\sin(\alpha-\beta)\sin(\gamma-\delta)\sin(\alpha+\beta)\sin(\gamma+\delta)=0\ldots(b),$$

for from (a) we deduce that

$$(\beta-\gamma)(\alpha-\delta)+(\gamma-\alpha)(\beta-\delta)+(\alpha-\beta)(\gamma-\delta)=0,$$

* This curious formula was given by Mr. C. H. Prior in the Mathematical Tripos examination for 1878 (Wednesday morning, January 2, Question 4: see *Solutions of the Cambridge Senate-House Problems and Riders for* 1878, p. 80).

i.e. writing β^2, γ^2, ... for β, γ, ..., that

$$(\beta^2-\gamma^2)(\alpha^2-\delta^2)+(\gamma^2-\alpha^2)(\beta^2-\delta^2)+(\alpha^2-\beta^2)(\gamma^2-\delta^2)=0,$$

which gives (*b*) on replacing β, γ, ... by $\sin\beta$, $\sin\gamma$, ...

The result (*b*) is in fact (*B*), which is thus seen to be an immediate corollary from the well-known formula (*a*).

Similarly from

$$\begin{aligned}&\sin(\beta-\gamma)\sin(\beta-\delta)\sin(\gamma-\delta)\\&+\sin(\gamma-\alpha)\sin(\gamma-\delta)\sin(\alpha-\delta)\\&+\sin(\alpha-\beta)\sin(\alpha-\delta)\sin(\beta-\delta)\\&+\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta)=0\ \ldots\ldots\ (c),\end{aligned}$$

we infer that

$$\begin{aligned}&\sin(\beta-\gamma)\sin(\beta-\delta)\sin(\gamma-\delta)\sin(\beta+\gamma)\sin(\beta+\delta)\sin(\gamma+\delta)\\&+\sin(\gamma-\alpha)\sin(\gamma-\delta)\sin(\alpha-\delta)\sin(\gamma+\alpha)\sin(\gamma+\delta)\sin(\alpha+\delta)\\&+\sin(\alpha-\beta)\sin(\alpha-\delta)\sin(\beta-\delta)\sin(\alpha+\beta)\sin(\alpha+\delta)\sin(\beta+\delta)\\&+\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta)\sin(\beta+\gamma)\sin(\gamma+\alpha)\sin(\alpha+\beta)\\&=0,\end{aligned}$$

which is the formula obtained in § 8.

§ 13. By equating terms of the fourth order in the expansion of the sines in (*a*), we have

$$\begin{aligned}&(\beta-\gamma)(\alpha-\delta)\{(\beta-\gamma)^2+(\alpha-\delta)^2\}\\&+(\gamma-\alpha)(\beta-\delta)\{(\gamma-\alpha)^2+(\gamma-\delta)^2\}\\&+(\alpha-\beta)(\gamma-\delta)\{(\alpha-\beta)^2+(\gamma-\delta)^2\}=0,\end{aligned}$$

from which we infer that

$$\begin{aligned}&\sin(\beta-\gamma)\sin(\alpha-\delta)\sin(\beta+\gamma)\sin(\alpha+\delta)\\&\quad\times\{\sin^2(\beta-\gamma)\sin^2(\beta+\gamma)+\sin^2(\alpha-\delta)\sin^2(\alpha+\delta)\}\\&+\sin(\gamma-\alpha)\sin(\beta-\delta)\sin(\gamma+\alpha)\sin(\beta+\delta)\\&\quad\times\{\sin^2(\gamma-\alpha)\sin^2(\gamma+\alpha)+\sin^2(\beta-\delta)\sin^2(\beta+\delta)\}\\&+\sin(\alpha-\beta)\sin(\gamma-\delta)\sin(\alpha+\beta)\sin(\gamma+\delta)\\&\quad\times\{\sin^2(\alpha-\beta)\sin^2(\alpha+\beta)+\sin^2(\gamma-\delta)\sin^2(\gamma+\delta)\}=0,\end{aligned}$$

and a corresponding result may be deduced from (*c*).

The arguments in the equation last written may of course be expressed in terms of $a, b \ldots a', \ldots a'', \ldots$ by means of the equations in § 1, connecting these quantities and $\alpha+\beta$, $\alpha-\beta$, ...

NOTE ON A THEOREM RELATING TO QUADRATIC EXPRESSIONS.

By Professor *H. Lamb, M.A.*

In § 319 [Example (*G*)] of the new edition of Thomson and Tait's *Natural Philosophy* there is given an important extension of Lagrange's generalized equations of motion. A part of the investigation consists in the proof of an algebraical theorem, which may be stated as follows:

If T be a homogeneous quadratic function of the $m+n$ variables $x_1, x_2, \ldots x_m, y_1, y_2, \ldots y_n$, and if the variables y be eliminated by means of n relations of the form

$$\eta = \frac{dT}{dy},$$

then T will appear as the sum of two homogeneous quadratic functions of the x's and the η's respectively; in other words, products of x's into η's will not occur in the result.

The following somewhat simplified proof of the theorem may perhaps be of interest.

We may write

$$T = F(x) + G(x, y) + H(y),$$

where F, H are homogeneous quadratic functions of the x's and the y's respectively, and G consists of products of x's into y's. Let

$$\eta' = \frac{dH}{dy} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and

$$\eta'' = \frac{dG}{dy} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

so that

$$\eta = \eta' + \eta''.$$

Then, if $K(\eta')$ denote the function of η', which $H(y)$ becomes on elimination of the y's by means of the n relations (1), we have, by a known theorem,

$$y = \frac{dK(\eta')}{d\eta'}.$$

Hence

$$\begin{aligned} G(x, y) + H(y) &= \Sigma y \frac{dG}{dy} + K(\eta') \\ &= \Sigma \eta'' \frac{dK(\eta')}{d\eta'} + K(\eta') \\ &= K(\eta + \eta'') - K(\eta'') \\ &= K(\eta) - K(\eta''). \end{aligned}$$

Here $K(\eta)$ denotes the same function of η that $K(\eta')$ is of η'.

Since the η'''s are linear functions of the x's, we may write, with Thomson and Tait,

$$T = \mathfrak{T} + K,$$

where $T, = F(x) - K(\eta'')$, is a homogeneous quadratic function of the x's, and $K, = K(\eta)$, is a similar function of the η's.

April, 1880.

A THEOREM RELATING TO PFAFFIANS.

By Prof. Dr. *W. Zajackowski.*

On pp. 56–60 of vol VIII. of the *Messenger*, Mr. H. W. Lloyd Tanner has proved a theorem concerning pfaffians. In the postscript to this memoir Prof. Cayley states that this theorem follows from the properties of a skew determinant, the square of which is the given pfaffian, though the working out would, perhaps, be rather difficult. The above theorem being fundamental in the memoir of Mr. Tanner on the transformation of linear differential expressions (*Quarterly Journal*, Dec. 1878, pp. 34–64), I propose the following demonstration, which is not so difficult as Prof. Cayley seems to have expected, and can be made shorter and more symmetrical.

If we denote by P the pfaffian $(12, \ldots 2n)$, viz.

$$P = (12, \ldots 2n) = \sum_{i=2}^{i=2n} (1, i)(i+1, \ldots 2n, 2, \ldots i-1) \ldots (1),$$

where $(i, k) = -(k, i)$, and by $P_{i,k}$, $P_{i,j,k,l}$ ... the coefficients of (i, k), (i, j, k, l), ... in the pfaffian P, viz.

$$\left.\begin{aligned} P_{i,k} &= (-1)^{i+k}(1, \ldots i-1, i+1, \ldots k-1, k+1, \ldots 2n \\ P_{i,j,k,l} &= (-1)^{i+j+k+l}(1, \ldots i-1, i+1, \ldots j-1, j+1, \ldots \\ &\qquad k-1, k+1, \ldots l-1, l+1, \ldots 2n) \end{aligned}\right\} \ldots (2),$$

we have to prove that

$$\sum_{i=2}^{i=2m} P_{1i} \cdot P_{i+1, \ldots 2m, 2, \ldots i-1} = P \cdot P_{12, \ldots 2m} \ldots\ldots\ldots\ldots (3).$$

Let us consider the skew determinant S, the square of which is the proposed pfaffian

$$S = \Sigma \pm (1, 1)(2, 2) \ldots (2n, 2n) = P^2 \ldots\ldots\ldots (4),$$

and denote by $S_{i,k}$ the minor to (i, k) of this determinant.

As $(i, k) = -(k, i)$, $(i, i) = 0$, and S is of an even order, we have $S_{i,i} = 0$, $S_{i,k} = -S_{k,i}$.

Differentiating the identity (4) with respect to (i, k) and observing that

$$\frac{dS}{d(i, k)} = 2S_{i,k}, \quad \frac{dP}{d(i, k)} = P_{i,k},$$

we obtain

$$S_{i,k} = P.P_{i,k} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

But by a well-known property of determinants

$$\Sigma \pm S_{1,1} S_{2,2} \ldots S_{2m,2m} = S^{2m-1} \Sigma \pm (2m+1, 2m+1) \ldots (2n, 2n),$$

if $m \leqq n$.

Hence from (4) and (5), and from the relation

$$\Sigma \pm (2m+1, 2m+1) \ldots (2n, 2n) = (2m+1, 2m+2, \ldots 2n)^2 = P^2_{12, \ldots 2m}$$

it follows, dividing the last equation by P_{2m},

$$\Sigma \pm P_{1,1} . P_{2,2} \ldots P_{2m,2m} = P^{2m-2} . P^2_{12, \ldots 2m} \ldots\ldots (6).$$

The first member of this equation is a skew determinant and therefore equal to the square of a certain pfaffian; then, if we denote by accented numbers the subscripts of this new pfaffian, putting

$$(1', 2', 3', \ldots 2m') = P_{12}(3', \ldots 2m') + P_{13}(4', \ldots 2m', 2') + \ldots + P_{1,2m}\{2', 3', \ldots (2m-1)'\} \ldots\ldots\ldots\ldots (7),$$

and taking the square root of both numbers, we may write the equation (6) in the following manner:

$$(1', 2', \ldots 2m') = P^{m-1} . P_{1,2, \ldots 2m} \ldots\ldots\ldots\ldots\ldots (8).$$

Developing the left-hand number according to (7), we have

$$\sum_{i=2}^{i=2m} P_{1,i}\{(i+1)', \ldots 2m', 2', \ldots (i-1')\} = P^{m-1} . P_{1,2, \ldots 2m},$$

and as by (8)

$$\{(i+1)', 2m', 2', \ldots (i-1)'\} = P^{m-2} . P_{i+1, \ldots 2m, 2, \ldots i-1},$$

then, substituting this value and dividing by P^{m-2}, we obtain the required relation (3).

Lemberg (Austria),
April 27th, 1850.

ON THE INTEGRATION OF DIFFERENTIAL EQUATIONS.

By *W. H. L. Russell, F.R.S.*

In this paper I hope to point out that the methods I have already given for linear equations are also applicable to non-linear equations.

1. Let the equation be

$$(a+bx+cx^2)\frac{d^2u}{dx^2}+(a'+b'x+c'x^2)\frac{du}{dx}+(a''+b''x+c''x^2)u=0.$$

We will determine the condition that this equation may be satisfied by a rational and entire function of (x). Let

$$u=A_0+A_1x+\ldots+A_nx^n.$$

Then plainly $c''=0$, $n=-\dfrac{b''}{c}$, which must therefore be a positive integer. Also we have $(n+1)$ linear relations, by which we find A_{n-1} in terms of A_n, A_{n-2} in terms of A_{n-1} and A_n, and therefore in terms of A_n, the last of these equations being an equation of condition among the quantities $(a, b, c, a', \ldots)$. A_n is indeterminate, and may be regarded as the arbitrary constant.

2. Let us now take the non-linear equation

$$(a+bx+cx^2)\,y\frac{d^2y}{dx^2}+(a'+b'x+c'x^2)\frac{dy^2}{dx^2}+(a''+b''x+c''x^2)y=0,$$

and consider it in the same point of view.

Let $$y=A_0+A_1x\ldots+A_{n-1}x^{n-1}+A_nx^n,$$

then, substituting in the given equation, we have

$$\begin{aligned}&(a+bx+cx^2)\,(A_0+A_1x+\ldots+A_{n-1}x^{n-1}+A_nx^n)\\&\times\{2A_2+2.3.A_3x+\ldots+(n-1)(n-2)A_{n-2}x^{n-3}+n(n-1)A_nx^{n-2}\}\\&+(a'+b'x+c'x^2)\,(A_1+2A_2x+3A_3x^2+\ldots+nA_nx^{n-1})^2\\&+(a''+b''x+c''x^2)\,(A_0+A_1x+\ldots+A_nx^n)=0.\end{aligned}$$

Here the number of disposable constants is $(n+9)$ and the number of equations $(2n+1)$. Hence $2n+1$ must not

be greater than $(n+9)$, or (n) must not be greater than 8. Also we have $c(n-1)+nc'=0$, whence $n=\frac{c}{c+c'}$, which must therefore be a positive integer. We have (n) equations, by which $A_{n-1}, A_{n-2}, \ldots A_0$ can be successively determined in terms of A_n and n equations of condition, which determine A_n, and must be satisfied by the constants of the differential equation.

3. Let us now investigate the conditions that the differential equations may be satisfied by rational fractions, and commence with the linear equation

$$L\frac{d^n u}{dx^n}+M\frac{d^{n-1}u}{dx^{n-1}}+N\frac{d^{n-2}u}{dx^{n-2}}+\ldots=0,$$

when L, M, N are rational and entire functions of (x).

Let $y=\frac{P}{Q}$, where P and Q are also rational and entire functions of (x). Also let

$$Q=(x-a)^\mu(x-b)^\nu(x-c)^\rho\ldots,$$
$$q=(x-a)\ (x-b)\ (x-c)\ \ldots,$$

then it is easily seen that the result of substituting $y=\frac{P}{Q}$ in the equation must be of the form

$$LP_1+MP_2q+NP_3q^2+\ldots=0,$$

where P_1 contains none of the factors $(x-a)\ldots$. Hence q must be a factor of L, and therefore $a, b, c\ldots$ must be roots of the equation $L=0$.

It remains to determine the indices $\mu, \nu, \ldots$. Now put

$$y=\frac{A}{(x-a)^\mu}+\frac{B}{(x-a)^{\mu-1}}+\ldots,$$

then, substituting in the differential equation, we have an equation to determine μ. Take the highest integer values of (μ), and similarly the highest integer values of $\nu, \rho\ldots$; then, if the equation have a solution of the form $y=\frac{P}{Q}$ at all, it will be of the form

$$y=\frac{P}{(x-a)^\mu(x-b)^\nu(x-c)^\rho\ldots},$$

where $\mu, \nu, \rho \ldots$ are the values determined. Hence, substituting,

$$L_1 \frac{d^n P}{dx^n} + M_1 \frac{d^{n-1} P}{dx^{n-1}} + N_1 \frac{d^{n-2} P}{dx^{n-2}} + \ldots = 0$$

is an equation to determine P, a rational and entire function, by section 1.

(4) This solution is due to Liouville. I proceed to show that with certain modifications it will apply to non-linear equations. Let

$$L_1 \left(\frac{d^m y}{dx^m}\right)^r \left(\frac{d^n y}{dx^n}\right)^s + L_2 \left(\frac{d^{m'} y}{dx^{m'}}\right)^{r'} \left(\frac{d^{n'} y}{dx^{n'}}\right)^{s'} + L_3 y = 0.$$

Let now $y = \frac{P}{Q}$, where P and Q are as in the last section. Then, substituting in the differential equation,

$$\frac{L_1 P_1}{(x-a)^{(r+s)\mu + mr + sn} \ldots} + \frac{L_2 P_2}{(x-a)^{(r'+s')\mu + m'r' + n's'} \ldots} + \frac{L_3 P}{(x-a)^{\mu} \ldots} = 0.$$

Then $(x-a)$ must be a factor of L_1 if $(r+s)\mu + mr + sn$ is greater than $(r'+s')\mu + m'r' + s'n'$, and of L_2 in the contrary case, on the supposition, of course, that the equation admits of a solution of the form we are now considering; that is, $x-a$ must be a factor of L_1 or L_2, according as μ is greater or less than

$$\frac{m'r' + s'n' - mr - sn}{r+s-r'-s'}.$$

Hence, if $L_1 = (x-a_1)(x-b_1)\ldots$, $L_2 (x-a_2)(x-b_2)\ldots$, and we assume successively

$$y = \frac{A}{(x-a_1)^{\mu}} + \frac{B}{(x-b_1)^{\mu-1}} + \ldots,$$

$$y = \frac{A}{(x-b_1)^{\mu}} + \frac{B}{(x-b_1)^{\mu-1}} + \ldots,$$

$$y = \frac{A}{(x-a_2)^{\mu}} + \frac{B}{(x-a_2)^{\mu-1}} + \ldots \&c.,$$

determine μ by substitution in the equation, and select, in each case, the highest integer value of μ in accordance with the condition just obtained, we shall find

$$Q = (x-a_1)^{\mu_1} (x-b_1)^{\nu_1} \ldots (x-a_2)^{\mu_2} \ldots,$$

where the factors of Q are partly factors of L_1 and partly of L_2. Substituting then $y = \frac{P}{Q}$ in the differential equation, we have a non-linear differential equation for determining P a rational and entire function of (x).

The reason why we take the highest integer value for (μ), under conditions, is obvious. It is because, if $y = \frac{P}{(x-a)^{\mu'}\dots}$, where μ' is a lower integer value of (μ), the value of y might be written $\frac{P(x-a)^{\mu-\mu'}}{(x-a)^{\mu}}$, where $P(x-a)^{\mu-\mu'}$ is still a rational and entire function of (x). The same method would apply to non-linear equations with more terms. We should have to enquire what value of μ would render each of a series of indices $a\mu + b$, $a'\mu + b'$, ... the greatest.

5. Let us now consider the case of linear differential equations, when a solution is possible, of the form $y = Pe^{\int Q dx}$ where Q is a rational function of (x), and first let us consider the circumstances where Q is an entire function. Take the equation

$$(\alpha + \beta x)\frac{d^3u}{dx^3} + (\alpha' + \beta' x + \gamma' x^2)\frac{d^2u}{dx^2} + (\alpha'' + \beta'' x + \gamma'' x^2)\frac{du}{dx} + (\alpha''' + \beta''' x + \gamma''' x^2)u = 0.$$

Let us now put in this equation $y = Pe^{\int dx(\mu+\nu x)}$, we shall easily see that it is impossible for the exponential to contain higher powers of (x) than here given.

Then we shall have

$$\frac{dy}{dx} = \left\{\frac{dP}{dx} + (\mu + \nu x)P\right\} e^{\int dx(\mu+\nu x)},$$

$$\frac{d^2y}{dx^2} = \left\{\frac{d^2P}{dx^2} + 2(\mu + \nu x)\frac{dP}{dx} + (\mu + \nu x)^2 P + \nu P\right\} e^{\int dx(\mu+\nu x)},$$

$$\frac{d^3y}{dx^3} = \left\{\frac{d^3P}{dx^3} + 3(\mu + \nu x)\frac{d^2P}{dx^2} + 3(\mu + \nu x)^2\frac{dP}{dx} + (\mu + \nu x)^3 P + 3\nu\frac{dP}{dx} + 3\nu(\mu + \nu x)P\right\} e^{\int dx(\mu+\nu x)}.$$

Substituting these values in the differential equation and

equating the coefficients of the highest powers of (x) to zero we have $r^3\beta + r^2\gamma' = 0$, whence $r = -\frac{\gamma'}{\beta}$, and

$$\nu^3\alpha + 3\nu^2\mu\beta + 2\mu\nu\gamma' + \nu^2\beta' + \gamma''\nu = 0,$$

whence $$\mu = \frac{\alpha\gamma'}{\beta} + \frac{\gamma''}{\gamma'} - \frac{\beta'}{\beta}.$$

Hence μ and ν are invariants, that is, they are the same whatever the constants in P, and the same thing holds good whenever the equation, supposed of any degree with rational and entire coefficients, admits of a solution of the form $y = P\varepsilon^{\int Q dx}$, where Q is an entire function of the r^{th} degree.

The reason of this is obvious. It is because if we differentiate $P\varepsilon^{\int Q dx}$ m times, the two last terms of the coefficient of $\varepsilon^{\int Q dx}$ will be of the form $\frac{dP}{dx} Q^{m-1}$ and PQ^m, which, if P is of the s^{th} degree, are of the $s - 1 + (m-1)r$, and $s + mr$ degrees respectively, and, therefore, the degree of PQ^m is greater by $(r+1)$ than the degree of $\frac{dP}{dx} Q^{m-1}$; consequently the $(r+1)$ constants in Q can be determined independently of those in P.

6. The same thing holds good in non-linear differential equations.

Consider the equation

$$L\left(\frac{d^i y}{dx^i}\right)^p \left(\frac{d^m y}{dx^m}\right)^q \left(\frac{d^n y}{dx^n}\right)^r \ldots + M\left(\frac{d^{i'} y}{dx^{i'}}\right)^{p'} \left(\frac{d^{m'} y}{dx^{m'}}\right)^{q'} \left(\frac{d^{n'} y}{dx^{n'}}\right)^{r'} + \ldots = 0,$$

and let $y = P\varepsilon^{\int Q dx}$ as before, where Q is a rational and entire function of (x).

Then it is easily seen that if

$$p + q + r \ldots = p' + q' + r' \ldots = \&c.,$$

then Q may be determined as before independently of the constants in P.

But if $p + q + r \ldots$, $p' + q' + r' \ldots$ &c. are not equal to each other, we have an equation of the form

$$R_0 + R_1\varepsilon^{\int Q dx} + R_2\varepsilon^{2\int Q dx} + \ldots = 0,$$

whence $$R_0 = 0,\ R_1 = 0,\ R_2 = 0 \ldots$$

in each of these equations Q will be determined independently of the constants in P; hence, we have a number of equations

of condition between the constants in the coefficients of the proposed differential equation.

7. No linear differential equation can have a solution of the form $y=f(\varepsilon^x)$. For let

$$\phi_n x\frac{d^n y}{dx^n}+\phi_{n-1}x\frac{d^{n-1}y}{dx^{n-1}}+\phi_{n-2}x\frac{d^{n-2}y}{dx^{n-2}}+\ldots=0,$$

and substitute in this equation $x+2\pi i$, $x+4\pi i\ldots$ successively for (x). This will give rise to a series of equations of the form

$$\phi_n(x)\quad\frac{d^n f(\varepsilon^x)}{dx^n}+\phi_{n-1}(x)\quad\frac{d^{n-1}f(\varepsilon^x)}{dx^{n-1}}+\ldots=0,$$

$$\phi_n(x+2\pi i)\frac{d^n f(\varepsilon^x)}{dx^n}+\phi_{n-1}(x+2\pi i)\frac{d^{n-1}f\varepsilon^x}{dx^{n-1}}+\ldots=0,$$

$$\phi_n(x+4\pi i)\quad\frac{d^n f\varepsilon^x}{dx^n}+\phi_{n-1}(x+4\pi i)\frac{d^{n-1}f\varepsilon^x}{dx^{n-1}}+\ldots=0.,$$

and these equations can be indefinitely continued.

9. For non linear equations this does not hold good, but in place of this theorem we have another as follows.

Let $y=\dfrac{p+q\varepsilon^{\int udx}}{r+s\varepsilon^{\int udx}}$ be the solution of a non linear equation, then expanding y,

$$y=u_0+u_1\varepsilon^{\int udx}+u_2\varepsilon^{2\int udx}+\ldots$$

and substituting this in the proposed non linear equation, we obtain an equation of the form

$$V_0+V_1\varepsilon^{\int udx}+V_2\varepsilon^{2\int udx}+\ldots=0,$$

when V_0. V_1, V_2, ... must all equal zero.

It follows from this that $y=\dfrac{p}{r}$ must be a solution of the given non linear equations.

Thus the equation $\dfrac{dy}{dx}=xy^2-y^2-y$ is satisfied by $y=\dfrac{1}{x+\varepsilon^x}$, and it is also satisfied by $y=\dfrac{1}{x}$.

We see moreover that (u) may be determined irrespectively of the constants in p, q, r, s.

10. Precisely similar reasoning applies to solutions of the form

$$y=\frac{p+q\,\varepsilon^{\int u dx}+r\,\varepsilon^{\int v dx}}{p_1+q_1\varepsilon^{\int u dx}+r_1\varepsilon^{\int v dx}},$$

where $p_1, q_1, r, p_1, q_1, r_1$, are rational and entire functions of (x), and (u) and (v) are the same whatever the constants in those functions.

In this and the preceding section (u) and (v) are assumed to be rational and entire functions of (x). When they are not entire functions of (x), the solution becomes much more complicated, and for the present I reserve it.

(To be continued.)

THE SYMMETRIC FUNCTIONS OF THE ROOTS OF AN EQUATION.

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1. THE equation is supposed to be

$$(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)\ldots=0,$$

or

$$x^n+bx^{n-1}+cx^{n-2}+dx^{n-3}+\ldots\ldots=0,$$

where n is greater than the degree of any symmetric function considered. Let the root α be called prior to β, β to γ, γ to δ, ..., so that α is prior to all the roots, β prior to all but α, γ to all but α and β, and so on. Then in writing down the typical term of any symmetric function, that term is chosen in which the power of a root is not greater than that of another prior to it; thus, in the order 7, $\alpha^4\beta^2\gamma$ would be chosen in preference to $\alpha^4\beta\gamma^2$ or $\alpha^3\beta^4\gamma$. Again, in making up the term, only those roots prior to all the others are used; thus $\alpha^4\beta^2\gamma$ is taken in preference to $\beta^4\varepsilon^2\zeta$ or any other combination.

This method of representing the typical term allows all the symmetric functions of any degree to be written down with ease. Thus, taking any definite power of α, and assigning to β powers beginning from unity and increasing by 1, all the functions containing the fixed power of α and

the assigned power of β are written down: giving β all permissible powers, the functions for the assumed power of α are obtained. Assigning to α all values from unity to the number representing the degree in question, we have the system of functions of this degree, the functions being always formed in accordance with the above restriction. Thus, in case of the degree 13, the functions containing x^5 are

$$\Sigma\alpha^5\beta\gamma\delta\epsilon\zeta\eta\theta, \quad \Sigma\alpha^5\beta^2\gamma\delta\epsilon\zeta, \quad \Sigma\alpha^5\beta^2\gamma\delta\epsilon, \quad \Sigma\alpha^5\beta^2\gamma^2,$$
$$\Sigma\alpha^5\beta^2\gamma\delta\epsilon\zeta\eta, \quad \Sigma\alpha^5\beta^2\gamma^2\delta\epsilon, \quad \Sigma\alpha^5\beta^2\gamma^2\delta\zeta, \quad \Sigma\alpha^5\beta^3\gamma,$$
$$\Sigma\alpha^5\beta^2\gamma^2\delta\epsilon\zeta, \quad \Sigma\alpha^5\beta^2\gamma^2\delta^2, \quad \Sigma\alpha^5\beta^3\gamma^2.$$
$$\Sigma\alpha^5\beta^3\gamma^2\delta\epsilon, \quad \Sigma\alpha^5\beta^3\gamma^3\zeta, \quad \Sigma\alpha^5\beta^4\gamma\delta.$$

2. In the following a formula is obtained from which the number of functions for any degree can be calculated: a table is given for the numbers in the degrees 1 to 31: and a method of calculating the functions in terms of the coefficients of the equation from those of the degree next below is indicated.

3. Let the number of symmetric functions of the degree r containing α be denoted by $\overset{r}{P}_s$; the whole number of that degree is evidently $\sum_1^r (s)\, \overset{r}{P}_s$.

If s be greater than $\frac{1}{2}r$ (r even),
or ..... $\frac{1}{2}(r-1)$ (r odd), then

$$\overset{r}{P}_s = \overset{r-1}{P}_{s-1} \quad \text{.....(i).}$$

If s be not greater than $\frac{1}{2}r$ (r even) or $\frac{1}{2}(r-1)$ (r odd), then

$$\overset{r}{P}_s - \overset{r-1}{P}_{s-1} = \text{number of functions containing } \alpha^s\beta^s$$
$$= \overset{r-s}{P}_s \quad \text{.....(ii)},$$

where on the right-hand side in $\overset{r-s}{P}_s$ only $n-1$ roots are employed; but this affects the number of terms in any assigned function, not the number of functions.

First, let r be even. Then by (i)

$$\sum_{\frac{1}{2}r+1}^{r} (s)\, \overset{r}{P}_s = \sum_{\frac{1}{2}r+1}^{r} (s)\, \overset{r-1}{P}_{s-1} = \sum_{\frac{1}{2}r}^{r-1} (s)\, \overset{r-1}{P}_s;$$

by (ii)
$$\begin{aligned}
{}^{r}_{\frac{1}{2}r}P &= {}^{r-1}_{\frac{1}{2}r-1}P + {}^{\frac{1}{2}r}_{\frac{1}{2}r}P,\\
{}^{r}_{\frac{1}{2}r-1}P &= {}^{r-1}_{\frac{1}{2}r-2}P + {}^{\frac{1}{2}r+1}_{\frac{1}{2}r-1}P,\\
&\cdots\cdots\cdots\cdots\\
{}^{r}_{2}P &= {}^{r-1}_{1}P + {}^{r-2}_{2}P
\end{aligned}$$

Therefore, adding,

$$\sum_{1}^{r}(s)\,{}^{r}_{s}P = \sum_{1}^{r-1}(s)\,{}^{r-1}_{s}P + {}^{r}_{1}P + {}^{\frac{1}{2}r}_{\frac{1}{2}r}P + {}^{\frac{1}{2}r+1}_{\frac{1}{2}r-1}P + {}^{\frac{1}{2}r+2}_{\frac{1}{2}r-2}P + \ldots + {}^{r-3}_{3}P + {}^{r-2}_{2}P \ldots \text{(I)}.$$

Second, let r be odd. Then, in exactly the same manner,

$$\sum_{1}^{r}(s)\,{}^{r}_{s}P = \sum_{1}^{r-1}(s)\,{}^{r-1}_{s}P + {}^{r}_{1}P + {}^{\frac{1}{2}(r+1)}_{\frac{1}{2}(r-1)}P + {}^{\frac{1}{2}(r+3)}_{\frac{1}{2}(r-3)}P + \ldots + {}^{r-3}_{3}P + {}^{r-2}_{2}P \ldots \text{(II)},$$

and ${}^{r}_{1}P$ is equal to unity.

Again, taking r even, so that $r-2$, $r-4$, ... are even, $r-1, r-3$, ... odd,

by (I) $$\sum_{1}^{r}(s)\,{}^{r}_{s}P - 1 = \sum_{1}^{r-1}(s)\,{}^{r-1}_{s}P + {}^{\frac{1}{2}r}_{\frac{1}{2}r}P + {}^{\frac{1}{2}r+1}_{\frac{1}{2}r-1}P + {}^{\frac{1}{2}r+2}_{\frac{1}{2}r-2}P + \ldots + {}^{r-4}_{4}P + {}^{r-3}_{3}P + {}^{r-2}_{2}P,$$

by (II) $$\sum_{1}^{r-1}(s)\,{}^{r-1}_{s}P - 1 = \sum_{1}^{r-2}(s)\,{}^{r-2}_{s}P + {}^{\frac{1}{2}r}_{\frac{1}{2}r-1}P + {}^{\frac{1}{2}r+1}_{\frac{1}{2}r-2}P + {}^{\frac{1}{2}r+2}_{\frac{1}{2}r-3}P + \ldots + {}^{r-4}_{3}P + {}^{r-3}_{2}P,$$

by (I) $$\sum_{1}^{r-2}(s)\,{}^{r-2}_{s}P - 1 = \sum_{1}^{r-3}(s)\,{}^{r-3}_{s}P + {}^{\frac{1}{2}r}_{\frac{1}{2}r-2}P + {}^{\frac{1}{2}r+1}_{\frac{1}{2}r-3}P + {}^{\frac{1}{2}r+2}_{\frac{1}{2}r-4}P + \ldots + {}^{r-4}_{2}P + {}^{\frac{1}{2}r-1}_{\frac{1}{2}r-1}P,$$

by (II) $$\sum_{1}^{r-3}(s)\,{}^{r-3}_{s}P - 1 = \sum_{1}^{r-4}(s)\,{}^{r-4}_{s}P + {}^{\frac{1}{2}r}_{\frac{1}{2}r-3}P + {}^{\frac{1}{2}r+1}_{\frac{1}{2}r-4}P + \ldots\ldots\ldots\ldots + {}^{\frac{1}{2}r-1}_{\frac{1}{2}r-2}P,$$

by (I) $$\sum_{1}^{r-4}(s)\,{}^{r-4}_{s}P - 1 = \sum_{1}^{r-5}(s)\,{}^{r-5}_{s}P + {}^{\frac{1}{2}r}_{\frac{1}{2}r-4}P + {}^{\frac{1}{2}r+1}_{\frac{1}{2}r-5}P + \ldots\ldots\ldots + {}^{\frac{1}{2}r-2}_{\frac{1}{2}r-2}P + {}^{\frac{1}{2}r-1}_{\frac{1}{2}r-3}P,$$

$$\cdots\cdots\cdots\cdots

Adding these, and noticing that $\sum_{1}^{4}(s)\,\overset{4}{P}_{s}=5$, the number of functions of the degree 4, we have

$$\sum_{1}^{r}(s)\,\overset{r}{P}_{s}-(r-4)=5+\sum_{\frac{1}{2}r}^{r-2}(\mu)\sum_{2}^{r-\mu}(s)\,\overset{\mu}{P}_{s}+\sum_{3}^{\frac{1}{2}r-1}(\mu)\sum_{2}^{\mu}(s)\,\overset{\mu}{P}_{s},$$

or

$$\sum_{1}^{r}(s)\,\overset{r}{P}_{s}=r+1+\sum_{\frac{1}{2}r}^{r-2}(\mu)\sum_{2}^{r-\mu}(s)\,\overset{\mu}{P}_{s}+\sum_{3}^{\frac{1}{2}r-1}(\mu)\sum_{2}^{\mu}(s)\,\overset{\mu}{P}_{s}\ldots(r\text{ even})$$

.....(III).

In the same way, taking r an odd integer,

$$\sum_{1}^{r}(s)\,\overset{r}{P}_{s}=r+1+\sum_{\frac{1}{2}(r+1)}^{r-2}(\mu)\sum_{2}^{r-\mu}(s)\,\overset{\mu}{P}_{s}+\sum_{3}^{\frac{1}{2}(r-1)}(\mu)\sum_{2}^{\mu}(s)\,\overset{\mu}{P}_{s}\ldots(r\text{ odd})\ldots(\text{IV}).$$

Both (III) and (IV) may be included in one formula. If p denote the integral part of $\frac{1}{2}(r+1)$, then

$$\sum_{1}^{r}(s)\,\overset{r}{P}_{s}=r+1+\sum_{p}^{r-2}(\mu)\sum_{2}^{r-\mu}(s)\,\overset{\mu}{P}_{s}+\sum_{3}^{p-1}(\mu)\sum_{2}^{\mu}(s)\,\overset{\mu}{P}_{s}\ldots(\text{V}).$$

Writing u instead of $\sum_{1}^{r}(s)\,\overset{r}{P}_{s}$, we have, ($r$ even)

$$u_{r+1}-u_{r}=1+\sum_{\frac{1}{2}r+1}^{r-1}(\mu)\sum_{2}^{r+1-\mu}(s)\,\overset{\mu}{P}_{s}+\sum_{3}^{\frac{1}{2}r}(\mu)\sum_{2}^{\mu}(s)\,\overset{\mu}{P}_{s}-\sum_{\frac{1}{2}r}^{r-2}(\mu)\sum_{2}^{r-\mu}(s)\,\overset{\mu}{P}_{s}-\sum_{3}^{\frac{1}{2}r-1}(\mu)\sum_{2}^{\mu}(s)\,\overset{\mu}{P}_{s}$$

$$=1+\sum_{\frac{1}{2}r+1}^{r-1}(\mu)\left(\overset{\mu}{P}_{2}+\overset{\mu}{P}_{3}+\ldots+\overset{\mu}{P}_{r+1-\mu}\right)-\sum_{\frac{1}{2}r}^{r-2}(\mu)\left(\overset{\mu}{P}_{2}+\overset{\mu}{P}_{3}+\ldots+\overset{\mu}{P}_{r-\mu}\right)+\sum_{2}^{\frac{1}{2}r}(s)\,\overset{\frac{1}{2}r}{P}_{s}$$

$$=1+\overset{r-1}{P}_{2}+\overset{r-2}{P}_{3}+\overset{r-3}{P}_{4}+\ldots+\overset{\frac{1}{2}r+1}{P}_{\frac{1}{2}r}-\sum_{2}^{\frac{1}{2}r}(s)\,\overset{\frac{1}{2}r}{P}_{s}+\sum_{2}^{\frac{1}{2}r}(s)\,\overset{\frac{1}{2}r}{P}_{s}$$

$$=\overset{r}{P}_{1}+\overset{r-1}{P}_{2}+\overset{r-2}{P}_{3}+\overset{r-3}{P}_{4}+\ldots+\overset{\frac{1}{2}r+1}{P}_{\frac{1}{2}r}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{VI}).$$

Similarly, (r odd)

$$u_{r+1}-u_{r}=\overset{r}{P}_{1}+\overset{r-1}{P}_{2}+\overset{r-2}{P}_{3}+\ldots+\overset{\frac{1}{2}(r+1)}{P}_{\frac{1}{2}(r+1)}\ldots\ldots\ldots\ldots(\text{VII}),$$

both of which are included in the form

$$u_{r+1}-u_{r}=\sum_{1}^{p}(s)\,\overset{r-s+1}{P}_{s},$$

p being the integral part of $\frac{1}{2}(r+1)$.

4. The following table was calculated from (VI) and (VII). The calculation was shortened by using the formula

$$u_{m-l} = {}^{m}P_{l},$$

if l be greater than $\frac{1}{2}m$, and, after three or four had been calculated, by the use of

$${}^{k}P_{\lambda} = {}^{k-1}P_{\lambda-1} + {}^{k-\lambda}P_{\lambda},$$

when 2λ is less than k or $k-1$. In actually evaluating any one

$${}^{r}P_{s} = \quad 1 \qquad\qquad \text{due to } \alpha^{r}\beta\gamma\delta\ldots$$

$$+\text{integer in } \tfrac{1}{2}(r-s) \quad \ldots\ldots \quad \alpha^{r}\beta^{s}\ldots$$
$$+1 \quad \ldots\ldots \quad \alpha^{r}\beta^{s}\gamma\delta\ldots$$
$$+\text{integer in } \tfrac{1}{2}(r-s-3) \quad \ldots\ldots \quad \alpha^{r}\beta^{s}\gamma^{2}\delta^{2}\ldots$$
$$+1 \quad \ldots\ldots \quad \alpha^{r}\beta^{s}\gamma^{3}\delta\ldots$$
$$+\text{integer in } \tfrac{1}{2}(r-s-6) \quad \ldots\ldots \quad \alpha^{r}\beta^{s}\gamma^{3}\delta^{2}\varepsilon^{2}$$
$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+1 \quad \ldots\ldots \quad \alpha^{r}\beta^{4}\gamma\delta$$
$$+\text{integer in } \tfrac{1}{2}(r-s-4) \quad \ldots\ldots \quad \alpha^{r}\beta^{4}\gamma^{2}\delta^{2}$$
$$+1 \quad \ldots\ldots \quad \alpha^{r}\beta^{4}\gamma^{2}\delta^{4}\varepsilon$$
$$+\text{integer in } \tfrac{1}{2}(r-s-7) \quad \ldots\ldots \quad \alpha^{r}\beta^{4}\gamma^{3}\delta^{2}\varepsilon^{2}$$
$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= 1+\text{integer in } \tfrac{1}{2}(r-s) \quad +1+\text{integer in } \tfrac{1}{2}(r-s-3) \quad +\ldots$$
$$+1+\ \ldots\ldots\ldots\ \tfrac{1}{2}(r-s-4) \quad +1+\ \ldots\ldots\ldots\ \tfrac{1}{2}(r-s-7) \quad +\ldots$$
$$+1+\ \ldots\ldots\ldots\ \tfrac{1}{2}(r-s-8) \quad +1+\ \ldots\ldots\ldots\ \tfrac{1}{2}(r-s-11)+\ldots$$
$$+1+\ \ldots\ldots\ldots\ \tfrac{1}{2}(r-s-12)+1+\ \ldots\ldots\ldots\ \tfrac{1}{2}(r-s-15)+\ldots$$
$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+1+\ \ldots\ldots\ldots\ \tfrac{1}{2}(r-s-5) \quad +1+\ \ldots\ldots\ldots\ \tfrac{1}{2}(r-s-8) \quad +\ldots$$
$$+1+\ \ldots\ldots\ldots\ \tfrac{1}{2}(r-s-9) \quad +1+\ \ldots\ldots\ldots\ \tfrac{1}{2}(r-s-12)+\ldots$$
$$+1+\ \ldots\ldots\ldots\ \tfrac{1}{2}(r-s-13)+1+\ \ldots\ldots\ldots\ \tfrac{1}{2}(r-s-16)+\ldots$$
$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+1+\ \ldots\ldots\ldots\ \tfrac{1}{2}(r-s-10)+1+\ \ldots\ldots\ldots\ \tfrac{1}{2}(r-s-13)+\ldots$$
$$+1+\ \ldots\ldots\ldots\ \tfrac{1}{2}(r-s-14)+1+\ \ldots\ldots\ldots\ \tfrac{1}{2}(r-s-17)+\ldots$$
$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

As an example,

$$u_{13}-u_{12}=1+P_2^{11}+P_3^{10}+P_4^{9}+P_5^{8}+P_6^{7}$$
$$=1+1+4+1+3+1+2+3+1=24$$
$$+1+2+1+1$$
$$+1\quad+1,$$

and $u_{12}=77$; therefore $u_{13}=101$.

The following is the table:

Degree.	Number of Functions.	Degree.	Number of Functions.	Degree.	Number of Functions.
1	1	11	56	21	792
2	2	12	77	22	1001
3	3	13	101	23	1254
4	5	14	135	24	1573
5	7	15	176	25	1955
6	11	16	231	26	2432
7	15	17	297	27	3005
8	22	18	385	28	3711
9	30	19	490	29	4557
10	42	20	627	30	5595

checked by means of

$$u_r-u_{r-1}-u_{r-2}+u_{r-4}+u_{r-5}+u_{r-6}=P_2^{r-4}+P_3^{r-9}+P_4^{r-12}+P_5^{r-15}+P_6^{r-18}+\ldots$$

continued so long as the upper suffix of P is not less than the lower suffix.

5. The formulæ (VI.), (VII.) may be put into a simpler analytical form; but it is less suitable for numerical calculation; we have

$$u_{2p+1}-u_{2p}=P_1^{2p}+P_2^{2p-1}+P_3^{2p-2}+\ldots+P_{p-1}^{p+2}+P_p^{p+1},$$

$$u_{2p}-u_{2p-1}=P_1^{2p-1}+P_2^{2p-2}+\ldots+P_{p-2}^{p+2}+P_{p-1}^{p+1}+P_p^{p},$$

$$u_{2p-1}-u_{2p-2}=P_1^{2p-2}+\ldots+P_{p-3}^{p+2}+P_{p-2}^{p+1}+P_{p-1}^{p},$$

$$u_{2p-2}-u_{2p-3}=P_1^{2p-3}+\ldots+P_{p-2}^{p}+P_{p-1}^{p-1},$$

.....

$$\begin{aligned} u_6 - u_5 &= P_1^5 + P_2^4 + P_3^3, \\ u_5 - u_4 &= \quad P_1^4 + P_2^3, \\ u_4 - u_3 &= \quad\quad P_1^3 + P_2^2, \\ u_3 - u_2 &= \quad\quad\quad P_1^2. \end{aligned}$$

Therefore, adding,

$$u_{2p+1} - u_2 = \overset{2p}{P}_1 + \ldots + \sum_1^{s+1} (\sigma) \overset{2p-s}{P}_\sigma + \ldots + \sum_1^{p} (\sigma) \overset{p+1}{P}_\sigma + \sum_2^{p} (\sigma) \sum_1^{\mu} (\mu) \overset{\mu}{P}_\sigma.$$

But $$\overset{3p+1}{P}_p = \sum_1^{p} (\sigma) \overset{2p+1}{P}_\sigma$$

$$= \overset{2p}{P}_1 + \sum_1^{2} (\sigma) \overset{2p-1}{P}_\sigma + \sum_1^{3} (\sigma) \overset{2p-2}{P}_\sigma + \ldots + \sum_1^{s+1} (\sigma) \overset{2p-s}{P}_\sigma + \ldots + \sum_1^{p} (\sigma) \overset{p+1}{P}_\sigma.$$

Therefore

$$u_{2p+1} = u_2 + \overset{3p+1}{P}_p + \sum_2^{p} (\mu) \sum_1^{\mu} (\sigma) \overset{\mu}{P}_\sigma$$

$$= 1 + \overset{3p+1}{P}_p + \sum_1^{p} (\mu) \sum_1^{\mu} (\sigma) \overset{\mu}{P}_\sigma,$$

or

$$u_{2p+1} - \sum_1^{p} (\sigma) u_\sigma = 1 + \overset{3p+1}{P}_p \quad \ldots\ldots\ldots\ldots \text{(VIII)}.$$

This may be deduced *ab initio* by means of (ii). For

$$\begin{aligned} \overset{3p+1}{P}_p &= \overset{3p}{P}_{p-1} + \overset{2p+1}{P}_p, \\ \overset{3p}{P}_{p-1} &= \overset{3p-1}{P}_{p-2} + \overset{2p+1}{P}_{p-1}, \\ &\ldots\ldots\ldots\ldots \\ \overset{2p+3}{P}_2 &= \overset{2p+2}{P}_1 + \overset{2p+1}{P}_2; \end{aligned}$$

therefore

$$\overset{3p+1}{P}_p = \overset{2p+2}{P}_1 + \sum_2^{p} (\sigma) \overset{2p+1}{P}_\sigma$$

$$= 1 + u_{2p+1} - \overset{2p+1}{P}_1 - \sum_{p+1}^{2p+1} (\sigma) \overset{2p+1}{P}_\sigma$$

$$= u_{2p+1} - (u_p + u_{p-1} + \ldots + u_2 + u_1 + \overset{2p+1}{P}_{2p+1})$$

$$= u_{2p+1} - \sum_1^{p} (\sigma) u_\sigma - 1,$$

which is equivalent to (VIII).

Similarly we may obtain

$$u_{2p} - \sum_{1}^{p-1}{}_{\sigma}(\sigma)\, u = 1 + P_{p}^{3p} \quad \text{.....(IX).}$$

Also
$$u_{p-1} = P_{p}^{2p-1} = P_{p}^{3p-1} - P_{p-1}^{3p-2}$$
$$= P_{p}^{3p-1} - \{u_{2p-1} - \sum_{1}^{p-1}{}_{\sigma}(\sigma)\, u - 1\} \text{ by (VIII);}$$

therefore
$$u_{2p-1} - \sum_{1}^{p-2}{}_{\sigma}(\sigma)\, u = 1 + P_{p}^{3p-1} \quad \text{.....(X).}$$

By (VIII) and (IX),

$$u_{2p} - u_{2p-1} = P_{p}^{3p} - P_{p-1}^{3p-2};$$

so that if the right-hand side could be expressed in terms of $2p$, this difference equation would give the general value of u_r.

6. By the method of partitions, the number of functions of any degree in the roots is equal to the number of functions of the same weight in the coefficients; and thus the latter is known for the weights 1 to 30 inclusive.

7. If Q denote the product of the n roots $\alpha\beta\gamma\delta\varepsilon\ldots$, and D the operator $\frac{d}{d\alpha} + \frac{d}{d\beta} + \frac{d}{d\gamma} + \ldots$, then

$$DQ = \alpha\beta\gamma\delta\ldots + \alpha\beta\delta\varepsilon\ldots + \ldots$$
$$= \text{sum of products taken } (n-1) \text{ at a time,}$$
$$D^2Q = 2.\text{sum}\ldots\ldots\ldots\ldots\ldots(n-2)\ldots\ldots\ldots,$$
$$D^3Q = 3.2.\text{sum}\ldots\ldots\ldots\ldots\ldots(n-3)\ldots\ldots\ldots,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$D^rQ = r!.\text{sum}\ldots\ldots\ldots\ldots\ldots(n-r)\ldots\ldots\ldots.$$

Substituting for these sums their values in terms of the coefficients of the equation, we obtain

$$\left.\begin{aligned} Db &= -n \\ Dc &= -(n-1)\, b \\ Dd &= -(n-2)\, c \\ D.\text{coeff. of } x^{n-r-1} &= -(n-r) \text{ coeff. of } x^{n-r} \end{aligned}\right\} \ldots(A).$$

Hence, if we have a relation between the coefficients and the roots, we may operate on both sides with D by means of (A) and obtain a relation of a lower degree; and, conversely, if the latter be known, unknown quantities occurring in the former may be determined. But in operating on a symmetric function care must be taken to insert the proper factors as they arise. Thus

$$D\Sigma\alpha^2 = 2\Sigma\alpha,$$

but
$$D\Sigma\alpha^2\beta = 2.2.\Sigma\alpha\beta + (n-1)\,\Sigma\alpha^2,$$

the additional factor 2 being introduced into the first term on the left-hand side because both $\alpha^2\beta$ and $\alpha\beta^2$ give rise to $\alpha\beta$. Similarly,

$$D\Sigma\alpha^7\beta^6\gamma^2\delta^2\varepsilon\zeta\eta\theta = 7.2.\Sigma\alpha^6\beta^6\gamma^2\delta^2\varepsilon\zeta\eta\theta + 6.\Sigma\alpha^7\beta^5\gamma^2\delta^2\varepsilon\zeta\eta\theta$$
$$+ 2.5.\Sigma\alpha^7\beta^6\gamma^2\delta\varepsilon\zeta\eta\theta + (n-7)\,\Sigma\alpha^7\beta^6\gamma^2\delta^2\varepsilon\zeta\eta,$$

the factor 5 being inserted in the third term because five times give rise to $\delta\varepsilon\zeta\eta\theta$. The number of terms containing $\alpha^7\beta^6\gamma^2\delta^2$ in $\Sigma\alpha^7\beta^6\gamma^2\delta^2\varepsilon\zeta\eta\theta$ is

$$\frac{(n-4)(n-5)(n-6)(n-7)}{1.2.3.4},$$

and each gives rise, through the operation of D, to 4 terms. The number of terms containing $\alpha^7\beta^6\gamma^2\delta^2$ in $\Sigma\alpha^7\beta^6\gamma^2\delta^2\varepsilon\zeta\eta$ is

$$\frac{(n-4)(n-5)(n-6)}{1.2.3}.$$

Hence each of these must occur altogether $n-7$ times; and therefore the factor $n-7$ is inserted in the fourth term of the left-hand side.

8. As an example, assuming the values of the symmetric functions of the tenth degree, suppose it be required to determine $\Sigma\alpha^2\beta^2\gamma^2\delta^2\varepsilon^2\zeta$. Denoting it by U, we have

$$DU = 2.2.\Sigma\alpha^2\beta^2\gamma^2\delta^2\varepsilon\zeta + (n-5)\,\Sigma\alpha^2\beta^2\gamma^2\delta^2\varepsilon^2$$
$$= 4\,(eg - 4dh + 9ci - 16bj + 25k)$$
$$+ (n-5)\,(f^2 - 2eg + 2dh - 2ci + 2bj - 2k) \ldots (1).$$

The only quantities which, when operated on with D, can give rise to the terms on the right-hand side are

$$fg,\ eh,\ di,\ cj,\ bk,\ l,\ f^2b,\ beg,\ bdh,\ bci,\ b^2j,$$

and of these the first six alone may be retained, because the others give rise to terms which do not occur in (1). Assuming

$$U = Afg + Beh + Cdi + Dcj + Ebk + Fl \ldots\ldots\ldots (2).$$

Substituting in (1) and equating coefficients, we have

$$\begin{aligned}
f^2 \ldots\ldots\ldots\ldots\ldots n-5 &= -(n-5)A,\\
eg \ldots\ldots 4-2(n-5) &= -(n-4)A-(n-6)B,\\
dh \ldots -16+2(n-5) &= -(n-3)B-(n-7)C,\\
ci \ldots\ldots 36-2(n-5) &= -(n-2)C-(n-8)D,\\
bj \ldots -64+2(n-5) &= -(n-1)D-(n-9)E,\\
k \ldots 100-2(n-5) &= -\quad n\quad E-(n-10)F,
\end{aligned}$$

which are satisfied by $A=-1$, $B=3$, $C=-5$, $D=7$, $E=-9$, $F=11$; and therefore

$$U = \Sigma\alpha^2\beta^2\gamma^2\delta^2\varepsilon^2\zeta = -fg + 3eh - 5di + 7cj - 9bk + 11l \ldots (3).$$

It is to be noticed that the values of the constants are independent of n, and this is so in every case; and the constants might be determined by equating coefficients of n.

Similarly, putting $U' = \Sigma\alpha^3\beta^2\gamma^2\delta^2\varepsilon^2$,

$$DU' = 3.5.\Sigma\alpha^2\beta^2\gamma^2\delta^2\varepsilon^2 + 2\Sigma\alpha^3\beta^2\gamma^2\delta^2\varepsilon.$$

Substituting and proceeding as before, we find

$$\Sigma\alpha^3\beta^2\gamma^2\delta^2\varepsilon^2 = -bf^2 + 2beg + fg - 2bdh - 3eh + 2bci + 5di + 11bk - 7cj - 2b^2j - 11l \ldots (4).$$

In the same way,

$$\Sigma\alpha^3\beta^3\gamma^2\delta^2\varepsilon\zeta = -beg + 4bdh + 5fg - 9bci + 16b^2j - 12eh + 10di - 70bk + 7cj + 110l \ldots (5).$$

By means of (3), (4), (5), we may deduce $\Sigma\alpha^3\beta^2\gamma^2\delta^2\varepsilon^2\zeta$. Denoting it by U'',

$$DU'' = 3.5.\Sigma\alpha^2\beta^2\gamma^2\delta^2\varepsilon^2\zeta + 2.2.\Sigma\alpha^3\beta^2\gamma^2\delta^2\varepsilon\zeta + (n-5)\,\Sigma\alpha^3\beta^2\gamma^2\delta^2\varepsilon^2 \ldots (6).$$

The result arrived at is

$$\Sigma\alpha^3\beta^2\gamma^2\delta^2\varepsilon^2\zeta = -6g^2 + bfg - 4ei + 10fh - 3beh + 5bdi - 6dj + 20ck - 7bcj + 9b^2k - 49bl + 60m \ldots (7).$$

9. This suggests an inductive proof of the rule for writing down the algebraical parts of the terms in any symmetrical function, no terms being retained containing a number of factors greater than the order of the function. Thus, for instance, the term b^2j occurs on the right-hand side of (6), and this might be derived from b^3j. But this, inserted in the trial value of U'', would give rise to a term b^3i which does not occur in (6); it is therefore omitted. According to the rule, it would be rejected because it contains four factors, whereas the order of the function is only three. The inductive proof of the rule is merely a generalization of the remarks applying to this particular case.

NOTE ON AN ALGEBRAICAL IDENTITY.

By *J. W. L. Glaisher.*

§1. THE present note relates to the theorem of Mr. C. H. Prior's, referred to on p. 33, viz. that if $a+b+c=0$, then

$$\left(\frac{b-c}{a}+\frac{c-a}{b}+\frac{a-b}{c}\right)\left(\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}\right)=9 \ldots \text{(i)}.$$

§2. The result (i) may be written in the form

$$\frac{bc(b-c)+ca(c-a)+ab(a-b)}{abc} \times \frac{a(c-a)(a-b)+b(a-b)(b-c)+c(b-c)(c-a)}{(b-c)(c-a)(a-b)}=9,$$

and, since

$$bc(b-c)+ca(c-a)+ab(a-b)=-(b-c)(c-a)(a-b),$$

this reduces to

$$a(a-b)(a-c)+b(b-c)(b-a)+c(c-a)(c-b)=9abc \ldots \text{(ii)},$$

which may be readily verified; for the left-hand side

$$\begin{aligned}&=\Sigma a\{a^2-a(b+c)+bc\}\\&=\Sigma a\{2a^2+bc\}, \text{ since } b+c=-a, \text{ \&c.}\\&=2(a^3+b^3+c^3)+3abc\\&=9abc, \text{ since } a^3+b^3+c^3=3abc.\end{aligned}$$

§3. If the expressions forming the left-hand member of (i) be multiplied together as they stand, the product

$$= 3 + \frac{\beta c}{\gamma b} + \frac{\gamma b}{\beta c} + \frac{\gamma a}{\alpha c} + \frac{\alpha c}{\gamma a} + \frac{\alpha b}{\beta a} + \frac{\beta a}{\alpha b}.$$

where for the moment α, β, γ denote $b-c$, $c-a$, $a-b$, so that (i) becomes

$$\frac{\beta^2 c^2 + \gamma^2 b^2}{\beta\gamma bc} + \frac{\gamma^2 a^2 + \alpha^2 c^2}{\gamma\alpha ca} + \frac{\alpha^2 b^2 + \beta^2 a^2}{\alpha\beta ab} = 6,$$

that is

$$\frac{(\beta c - \gamma b)^2}{\beta\gamma bc} + \frac{(\gamma a - \alpha c)^2}{\gamma\alpha ca} + \frac{(\alpha b - \beta a)^2}{\alpha\beta ab} = 0.$$

Now
$$\begin{aligned}\beta c - \gamma b &= c(c-a) - b(a-b)\\ &= c^2 + b^2 - a(b+c)\\ &= a^2 + b^2 + c^2,\end{aligned}$$

so that the formula becomes

$$\Sigma \frac{(a^2+b^2+c^2)^2}{bc(c-a)(a-b)} = 0,$$

that is
$$\Sigma \frac{(a^2+b^2+c^2)^2}{(c^2-a^2)(a^2-b^2)} = 0,$$

that is
$$(b^2-c^2) + (c^2-a^2) + (a^2-b^2) = 0.$$

§4. I now proceed to consider the value of the right-hand side of (i) when a, b, c are not subject to the condition $a+b+c=0$.

Proceeding as in §2, the left-hand side reduces as before to

$$\frac{a(a-b)(a-c) + b(b-c)(b-a) + c(c-a)(c-b)}{abc},$$

and it is easy to see that, identically,

$$\begin{aligned}&a(a-b)(a-c) + b(b-c)(b-a) + c(c-a)(c-b)\\ &\quad = 9abc + (a+b+c)(a^2+b^2+c^2-2bc-2ca-2ab) \ldots \text{(iii)},\end{aligned}$$

so that when a, b, c are unrestricted

$$\left(\frac{b-c}{a} + \frac{c-a}{b} + \frac{a-b}{c}\right)\left(\frac{a}{b-c} + \frac{b}{c-a} + \frac{c}{a-b}\right)$$
$$= 9 + \frac{(a+b+c)(a^2+b^2+c^2-2bc-2ca-2ab)}{abc} \quad \ldots\ldots \text{(iv.)}.$$

The second member of this identity reduces to its first term both when $a+b+c=0$ and when $\sqrt{a}+\sqrt{b}+\sqrt{c}=0$; putting a^2, b^2, c^2 for a, b, c (iv) may be written

$$\left(\frac{b^2-c^2}{a^2}+\frac{c^2-a^2}{b^2}+\frac{a^2-b^2}{c^2}\right)\left(\frac{a^2}{b^2-c^2}+\frac{b^2}{c^2-a^2}+\frac{c^2}{a^2-b^2}\right)=$$

$$9-\frac{(a^2+b^2+c^2)(a+b+c)-(a+b+c)(a-b+c)(a+b-c)}{a^2b^2c^2} \quad \ldots(\text{v}),$$

so that, if $a+b+c=0$,

$$\left(\frac{b-c}{a}+\frac{c-a}{b}+\frac{a-b}{c}\right)\left(\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}\right)=9,$$

and also

$$\left(\frac{b^2-c^2}{a^2}+\frac{c^2-a^2}{b^2}+\frac{a^2-b^2}{c^2}\right)\left(\frac{a^2}{b^2-c^2}+\frac{b^2}{c^2-a^2}+\frac{c^2}{a^2-b^2}\right)=9.$$

Each of these equations is however readily deducible from the other, for when $a+b+c=0$,

$$\frac{b-c}{a}=-\frac{(b+c)(b-c)}{a^2}=-\frac{b^2-c^2}{a^2}, \text{ \&c.}$$

§ 5. If a, b, c be the sides of a triangle, and A, B, C the opposite angles, then the second term on the right-hand side of (v)

$$=16S^2\frac{a^2+b^2+c^2}{a^2b^2c^2},$$

S denoting the area of the triangle,

$$=\frac{a^2+b^2+c^2}{R^2}=4\left(\sin^2A+\sin^2B+\sin^2C\right),$$

where R denotes the radius of the circumscribed circle.

Thus from (v) we obtain the trigonometrical formula

$$\left\{\frac{\sin(B-C)}{\sin A}+\frac{\sin(C-A)}{\sin B}+\frac{\sin(A-B)}{\sin C}\right\}$$

$$\times\left\{\frac{\sin A}{\sin(B-C)}+\frac{\sin B}{\sin(C-A)}+\frac{\sin C}{\sin(A-B)}\right\}$$

$$=9-4\left(\sin^2A+\sin^2B+\sin^2C\right)\ldots(\text{vi}).$$

From this we may also deduce the formula

$$\frac{\sin(A-B)\sin(A-C)}{\sin B\sin C}+\frac{\sin(B-C)\sin(B-A)}{\sin C\sin A}$$

$$+\frac{\sin(C-A)\sin(C-B)}{\sin A\sin B}=9-4(\sin^2A+\sin^2B+\sin^2C)\ldots\text{(vii)},$$

which corresponds to (iii).

It is easy to verify (vii), for

$$\sin A\sin(A-B)\sin(A-C)=\tfrac{1}{4}(\sin 2B+\sin 2C-\sin 2A+\sin 4A);$$

and therefore

$$\sin A\sin(A-B)\sin(A-C)+\sin B\sin(B-A)\sin(B-C)$$

$$+\sin C\sin(C-A)\sin(C-B)$$

$$=\tfrac{1}{4}(\sin 2A+\sin 2B+\sin 2C+\sin 4A+\sin 4B+\sin 4C)$$

$$=\sin A\sin B\sin C-\sin 2A\sin 2B\sin 2C$$

$$=\sin A\sin B\sin C(1-8\cos A\cos B\cos C)$$

$$=\sin A\sin B\sin C(9-8-8\cos A\cos B\cos C)$$

$$=\sin A\sin B\sin C(9-4\sin^2A-4\sin^2B-4\sin^2C).$$

§ 6. The formula (i), in which $a+b+c=0$, may also be written in the form

$$\left(\frac{b-c}{b+c}+\frac{c-a}{c+a}+\frac{a-b}{a+b}\right)\left(\frac{b+c}{b-c}+\frac{c+a}{c-a}+\frac{a+b}{a-b}\right)=9\ldots\text{(viii)},$$

and it seems a matter of interest to determine the value of the right-hand side, where a, b, c are not subject to the condition $a+b+c=0$.

Supposing a, b, c to be unrestricted, the left-hand side of (viii)

$$=\frac{(b-c)(c+a)(a+b)+(c-a)(a+b)(b+c)+(a-b)(b+c)(c+a)}{(b+c)(c+a)(a+b)}$$

$$\times\frac{(b+c)(c-a)(a-b)+(c+a)(a-b)(b-c)+(a+b)(b-c)(c-a)}{(b-c)(c-a)(a-b)}$$

which

$$=\frac{(b+c)(a-b)(a-c)+(c+a)(b-c)(b-a)+(a+b)(c-a)(c-b)}{(b+c)(c+a)(a+b)}\ldots(\alpha),$$

for

$$(b-c)(c+a)(a+b)+(c-a)(a+b)(b+c)+(a-b)(b+c)(c+a)$$

$$=-(b-c)(c-a)(a-b)\ldots\text{(ix)}.$$

4. The following table was calculated from (VI) and (VII). The calculation was shortened by using the formula

$$u_{m-l} = {}^{m}P_{l},$$

if l be greater than $\frac{1}{2}m$, and, after three or four had been calculated, by the use of

$${}^{k}P_{\lambda} = {}^{k-1}P_{\lambda-1} + {}^{k-\lambda}P_{\lambda},$$

when 2λ is less than k or $k-1$. In actually evaluating any one

$${}^{r}P_{s} = \begin{array}{ll} 1 & \text{due to } \alpha^s\beta\gamma\delta\ldots \\ +\text{integer in } \tfrac{1}{2}(r-s) & \ldots\ldots\ \alpha^s\beta^2\ldots \\ +1 & \ldots\ldots\ \alpha^s\beta^3\gamma\delta\ldots \\ +\text{integer in } \tfrac{1}{2}(r-s-3) & \ldots\ldots\ \alpha^s\beta^3\gamma^2\delta^2\ldots \\ +1 & \ldots\ldots\ \alpha^s\beta^3\gamma^3\delta\ldots \\ +\text{integer in } \tfrac{1}{2}(r-s-6) & \ldots\ldots\ \alpha^s\beta^3\gamma^3\delta^2\varepsilon^2 \\ +\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ +1 & \ldots\ldots\ \alpha^s\beta^4\gamma\delta \\ +\text{integer in } \tfrac{1}{2}(r-s-4) & \ldots\ldots\ \alpha^s\beta^4\gamma^2\delta^2 \\ +1 & \ldots\ldots\ \alpha^s\beta^4\gamma^3\delta^4\varepsilon \\ +\text{integer in } \tfrac{1}{2}(r-s-7) & \ldots\ldots\ \alpha^s\beta^4\gamma^3\delta^2\varepsilon^2 \\ +\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \end{array}$$

$$\begin{array}{ll} = & 1+\text{integer in } \tfrac{1}{2}(r-s) \quad +1+\text{integer in } \tfrac{1}{2}(r-s-3)\ +\ldots \\ & +1+\ \ldots\ldots\ldots\ \tfrac{1}{2}(r-s-4)\ +1+\ \ldots\ldots\ldots\ \tfrac{1}{2}(r-s-7)\ +\ldots \\ & +1+\ \ldots\ldots\ldots\ \tfrac{1}{2}(r-s-8)\ +1+\ \ldots\ldots\ldots\ \tfrac{1}{2}(r-s-11)+\ldots \\ & +1+\ \ldots\ldots\ldots\ \tfrac{1}{2}(r-s-12)+1+\ \ldots\ldots\ldots\ \tfrac{1}{2}(r-s-15)+\ldots \\ & +\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & +1+\ \ldots\ldots\ldots\ \tfrac{1}{2}(r-s-5)\ +1+\ \ldots\ldots\ldots\ \tfrac{1}{2}(r-s-8)\ +\ldots \\ & +1+\ \ldots\ldots\ldots\ \tfrac{1}{2}(r-s-9)\ +1+\ \ldots\ldots\ldots\ \tfrac{1}{2}(r-s-12)+\ldots \\ & +1+\ \ldots\ldots\ldots\ \tfrac{1}{2}(r-s-13)+1+\ \ldots\ldots\ldots\ \tfrac{1}{2}(r-s-16)+\ldots \\ & +\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & +1+\ \ldots\ldots\ldots\ \tfrac{1}{2}(r-s-10)+1+\ \ldots\ldots\ldots\ \tfrac{1}{2}(r-s-13)+\ldots \\ & +1+\ \ldots\ldots\ldots\ \tfrac{1}{2}(r-s-14)+1+\ \ldots\ldots\ldots\ \tfrac{1}{2}(r-s-17)+\ldots \\ & +\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

As an example,

$$u_{13} - u_{12} = 1 + P_2^{11} + P_3^{10} + P_4^9 + P_5^8 + P_6^7$$
$$= 1 + 1 + 4 + 1 + 3 + 1 + 2 + 3 + 1 = 24$$
$$+ 1 + 2 + 1 + 1$$
$$+ 1 \quad + 1,$$

and $u_{12} = 77$; therefore $u_{13} = 101$.

The following is the table:

Degree.	Number of Functions.	Degree.	Number of Functions.	Degree.	Number of Functions.
1	1	11	56	21	792
2	2	12	77	22	1001
3	3	13	101	23	1254
4	5	14	135	24	1573
5	7	15	176	25	1955
6	11	16	231	26	2432
7	15	17	297	27	3005
8	22	18	385	28	3711
9	30	19	490	29	4557
10	42	20	627	30	5595

checked by means of

$$u_r - u_{r-1} - u_{r-2} + u_{r-4} + u_{r-5} + u_{r-6} = P_2^{r-4} + P_3^{r-9} + P_4^{r-12} + P_5^{r-15} + P_6^{r-18} + \ldots$$

continued so long as the upper suffix of P is not less than the lower suffix.

5. The formulæ (VI.), (VII.) may be put into a simpler analytical form; but it is less suitable for numerical calculation; we have

$$u_{2p+1} - u_{2p} = P_1^{2p} + P_2^{2p-1} + P_3^{2p-2} + \ldots + P_{p-1}^{p+2} + P_p^{p+1},$$

$$u_{2p} - u_{2p-1} = P_1^{2p-1} + P_2^{2p-2} + \ldots + P_{p-2}^{p+2} + P_{p-1}^{p+1} + P_p^{p},$$

$$u_{2p-1} - u_{2p-2} = P_1^{2p-2} + \ldots + P_{p-3}^{p+2} + P_{p-2}^{p+1} + P_{p-1}^{p},$$

$$u_{2p-2} - u_{2p-3} = P_1^{2p-3} + \ldots + P_{p-2}^{p} + P_{p-1}^{p-1},$$

.....

$$\begin{aligned} u_6 - u_5 &= P_1^5 + P_2^4 + P_3^3, \\ u_5 - u_4 &= \quad P_1^4 + P_2^3, \\ u_4 - u_3 &= \quad\quad P_1^3 + P_2^2, \\ u_3 - u_2 &= \quad\quad\quad P_1^2. \end{aligned}$$

Therefore, adding,

$$u_{2p+1} - u_2 = \overset{2p}{P}_1 + \ldots + \sum_1^{s+1} (\sigma) \overset{2p-s}{P}_\sigma + \ldots + \sum_1^{p} (\sigma) \overset{p+1}{P}_\sigma + \sum_2^{p} (\sigma) \sum_1^{\mu} (\mu) \overset{\mu}{P}_\sigma.$$

But $$\overset{3p+1}{P}_p = \sum_1^{p} (\sigma) \overset{2p+1}{P}_\sigma$$

$$= \overset{2p}{P}_1 + \sum_1^{2} (\sigma) \overset{2p-1}{P}_\sigma + \sum_1^{3} (\sigma) \overset{2p-2}{P}_\sigma + \ldots + \sum_1^{s+1} (\sigma) \overset{2p-s}{P}_\sigma + \ldots + \sum_1^{p} (\sigma) \overset{p+1}{P}_\sigma.$$

Therefore

$$u_{2p+1} = u_2 + \overset{3p+1}{P}_p + \sum_2^{p} (\mu) \sum_1^{\mu} (\sigma) \overset{\mu}{P}_\sigma$$

$$= 1 + \overset{3p+1}{P}_p + \sum_1^{p} (\mu) \sum_1^{\mu} (\sigma) \overset{\mu}{P}_\sigma,$$

or

$$u_{2p+1} - \sum_1^{p} (\sigma)\, u_\sigma = 1 + \overset{3p+1}{P}_p \quad \ldots\ldots\ldots\ldots \text{(VIII)}.$$

This may be deduced *ab initio* by means of (ii). For

$$\begin{aligned} \overset{3p+1}{P}_p &= \overset{3p}{P}_{p-1} + \overset{2p+1}{P}_p, \\ \overset{3p}{P}_{p-1} &= \overset{3p-1}{P}_{p-2} + \overset{2p+1}{P}_{p-1}, \\ &\ldots\ldots\ldots\ldots \\ \overset{2p+3}{P}_2 &= \overset{2p+2}{P}_1 + \overset{2p+1}{P}_2; \end{aligned}$$

therefore

$$\overset{3p+1}{P}_p = \overset{2p+2}{P}_1 + \sum_2^{p} (\sigma) \overset{2p+1}{P}_\sigma$$

$$= 1 + u_{2p+1} - \overset{2p+1}{P}_1 - \sum_{p+1}^{2p+1} (\sigma) \overset{2p+1}{P}_\sigma$$

$$= u_{2p+1} - (u_p + u_{p-1} + \ldots + u_2 + u_1 + \overset{2p+1}{P}_{2p+1})$$

$$= u_{2p+1} - \sum_1^{p} (\sigma)\, u_\sigma - 1,$$

which is equivalent to (VIII).

Similarly we may obtain

$$u_{2p} - \sum_{1}^{p-1}{}_{\sigma}(\sigma)\, u = 1 + P_{p}^{3p} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(IX)}.$$

Also
$$u_{p-1} = P_{p}^{2p-1} = P_{p}^{3p-1} - P_{p-1}^{3p-2}$$
$$= P_{p}^{3p-1} - \{u_{2p-1} - \sum_{1}^{p-1}{}_{\sigma}(\sigma)\, u - 1\} \text{ by (VIII)};$$

therefore
$$u_{2p-1} - \sum_{1}^{p-2}{}_{\sigma}(\sigma)\, u = 1 + P_{p}^{3p-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(X)}.$$

By (VIII) and (IX),

$$u_{2p} - u_{2p-1} = P_{p}^{3p} - P_{p-1}^{3p-2};$$

so that if the right-hand side could be expressed in terms of $2p$, this difference equation would give the general value of u_r.

6. By the method of partitions, the number of functions of any degree in the roots is equal to the number of functions of the same weight in the coefficients; and thus the latter is known for the weights 1 to 30 inclusive.

7. If Q denote the product of the n roots $\alpha\beta\gamma\delta\varepsilon\ldots$, and D the operator $\frac{d}{d\alpha} + \frac{d}{d\beta} + \frac{d}{d\gamma} + \ldots$, then

$$DQ = \alpha\beta\gamma\delta\ldots + \alpha\beta\delta\varepsilon\ldots + \ldots$$
$$= \text{sum of products taken } (n-1) \text{ at a time},$$
$$D^2Q = 2.\text{sum}\ldots\ldots\ldots\ldots\ldots\ldots(n-2)\ldots\ldots\ldots,$$
$$D^3Q = 3.2.\text{sum}\ldots\ldots\ldots\ldots\ldots\ldots(n-3)\ldots\ldots\ldots,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$D^rQ = r!.\text{sum}\ldots\ldots\ldots\ldots\ldots\ldots(n-r)\ldots\ldots\ldots.$$

Substituting for these sums their values in terms of the coefficients of the equation, we obtain

$$\left.\begin{aligned} Db &= -n \\ Dc &= -(n-1)\, b \\ Dd &= -(n-2)\, c \\ D.\text{coeff. of } x^{n-r-1} &= -(n-r) \text{ coeff. of } x^{n-r} \end{aligned}\right\} \ldots(A).$$

Hence, if we have a relation between the coefficients and the roots, we may operate on both sides with D by means of (A) and obtain a relation of a lower degree; and, conversely, if the latter be known, unknown quantities occurring in the former may be determined. But in operating on a symmetric function care must be taken to insert the proper factors as they arise. Thus

$$D\Sigma\alpha^2 = 2\Sigma\alpha,$$

but
$$D\Sigma\alpha^2\beta = 2.2.\Sigma\alpha\beta + (n-1)\,\Sigma\alpha^2,$$

the additional factor 2 being introduced into the first term on the left-hand side because both $\alpha^2\beta$ and $\alpha\beta^2$ give rise to $\alpha\beta$. Similarly,

$$D\Sigma\alpha^7\beta^6\gamma^2\delta^2\varepsilon\zeta\eta\theta = 7.2.\Sigma\alpha^6\beta^6\gamma^2\delta^2\varepsilon\zeta\eta\theta + 6.\Sigma\alpha^7\beta^5\gamma^2\delta^2\varepsilon\zeta\eta\theta$$
$$+ 2.5.\Sigma\alpha^7\beta^6\gamma^2\delta\varepsilon\zeta\eta\theta + (n-7)\,\Sigma\alpha^7\beta^6\gamma^2\delta^2\varepsilon\zeta\eta,$$

the factor 5 being inserted in the third term because five times give rise to $\delta\varepsilon\zeta\eta\theta$. The number of terms containing $\alpha^7\beta^6\gamma^2\delta^2$ in $\Sigma\alpha^7\beta^6\gamma^2\delta^2\varepsilon\zeta\eta\theta$ is

$$\frac{(n-4)(n-5)(n-6)(n-7)}{1.2.3.4},$$

and each gives rise, through the operation of D, to 4 terms. The number of terms containing $\alpha^7\beta^6\gamma^2\delta^2$ in $\Sigma\alpha^7\beta^6\gamma^2\delta^2\varepsilon\zeta\eta$ is

$$\frac{(n-4)(n-5)(n-6)}{1.2.3}.$$

Hence each of these must occur altogether $n-7$ times; and therefore the factor $n-7$ is inserted in the fourth term of the left-hand side.

8. As an example, assuming the values of the symmetric functions of the tenth degree, suppose it be required to determine $\Sigma\alpha^2\beta^2\gamma^2\delta^2\varepsilon^2\zeta$. Denoting it by U, we have

$$DU = 2.2.\Sigma\alpha^2\beta^2\gamma^2\delta^2\varepsilon\zeta + (n-5)\,\Sigma\alpha^2\beta^2\gamma^2\delta^2\varepsilon^2$$
$$= 4\,(eg - 4dh + 9ci - 16bj + 25k)$$
$$+ (n-5)\,(f^2 - 2eg + 2dh - 2ci + 2bj - 2k) \ldots (1).$$

The only quantities which, when operated on with D, can give rise to the terms on the right-hand side are

$$fg,\ eh,\ di,\ cj,\ bk,\ l,\ f^2b,\ beg,\ bdh,\ bci,\ b^3j,$$

and of these the first six alone may be retained, because the others give rise to terms which do not occur in (1). Assuming

$$U = Afg + Beh + Cdi + Dcj + Ebk + Fl \ldots\ldots\ldots (2).$$

Substituting in (1) and equating coefficients, we have

$$\begin{aligned}
f^2 \ldots\ldots\ldots\ldots\ldots n-5 &= -(n-5)\,A,\\
eg \ldots\ldots 4-2(n-5) &= -(n-4)\,A-(n-6)\,B,\\
dh \ldots -16+2(n-5) &= -(n-3)\,B-(n-7)\,C,\\
ci \ldots\ldots 36-2(n-5) &= -(n-2)\,C-(n-8)\,D,\\
bj \ldots -64+2(n-5) &= -(n-1)\,D-(n-9)\,E,\\
k \ldots\ 100-2(n-5) &= -\quad n\quad E-(n-10)\,F,
\end{aligned}$$

which are satisfied by $A=-1$, $B=3$, $C=-5$, $D=7$, $E=-9$, $F=11$; and therefore

$$U = \Sigma\alpha^3\beta^2\gamma^2\delta^2\varepsilon^2\zeta = -fg + 3eh - 5di + 7cj - 9bk + 11l \ldots (3).$$

It is to be noticed that the values of the constants are independent of n, and this is so in every case; and the constants might be determined by equating coefficients of n.

Similarly, putting $U' = \Sigma\alpha^3\beta^3\gamma^2\delta^2\varepsilon^2$,

$$DU' = 3.5.\Sigma\alpha^2\beta^2\gamma^2\delta^2\varepsilon^2 + 2\Sigma\alpha^3\beta^2\gamma^2\delta^2\varepsilon.$$

Substituting and proceeding as before, we find

$$\Sigma\alpha^3\beta^2\gamma^2\delta^2\varepsilon^2 = -bf^2 + 2beg + fg - 2bdh - 3eh + 2bci + 5di + 11bk - 7cj - 2b^2j - 11l \ldots (4).$$

In the same way,

$$\Sigma\alpha^3\beta^2\gamma^2\delta^2\varepsilon\zeta = -beg + 4bdh + 5fg - 9bci + 16b^2j - 12eh + 10di - 70bk + 7cj + 110l \ldots (5).$$

By means of (3), (4), (5), we may deduce $\Sigma\alpha^3\beta^2\gamma^2\delta^2\varepsilon^2\zeta$. Denoting it by U'',

$$DU'' = 3.5.\Sigma\alpha^2\beta^2\gamma^2\delta^2\varepsilon^2\zeta + 2.2.\Sigma\alpha^3\beta^2\gamma^2\delta^2\varepsilon\zeta + (n-5)\,\Sigma\alpha^3\beta^2\gamma^2\delta^2\varepsilon^2 \ldots (6).$$

The result arrived at is

$$\Sigma\alpha^3\beta^2\gamma^2\delta^2\varepsilon^2\zeta = -6g^2 + bfg - 4ei + 10fh - 3beh + 5bdi - 6dj + 20ck - 7bcj + 9b^2k - 49bl + 60m \ldots (7).$$

9. This suggests an inductive proof of the rule for writing down the algebraical parts of the terms in any symmetrical function, no terms being retained containing a number of factors greater than the order of the function. Thus, for instance, the term b^2j occurs on the right-hand side of (6), and this might be derived from b^2j. But this, inserted in the trial value of U'', would give rise to a term b^3i which does not occur in (6); it is therefore omitted. According to the rule, it would be rejected because it contains four factors, whereas the order of the function is only three. The inductive proof of the rule is merely a generalization of the remarks applying to this particular case.

NOTE ON AN ALGEBRAICAL IDENTITY.

By *J. W. L. Glaisher.*

§1. THE present note relates to the theorem of Mr. C. H. Prior's, referred to on p. 33, viz. that if $a+b+c=0$, then

$$\left(\frac{b-c}{a}+\frac{c-a}{b}+\frac{a-b}{c}\right)\left(\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}\right)=9\ldots\text{(i)}.$$

§2. The result (i) may be written in the form

$$\frac{bc\,(b-c)+ca\,(c-a)+ab\,(a-b)}{abc}$$
$$\times\frac{a\,(c-a)\,(a-b)+b\,(a-b)\,(b-c)+c\,(b-c)\,(c-a)}{(b-c)\,(c-a)\,(a-b)}=9,$$

and, since

$$bc\,(b-c)+ca\,(c-a)+ab\,(a-b)=-\,(b-c)\,(c-a)\,(a-b),$$

this reduces to

$$a\,(a-b)\,(a-c)+b\,(b-c)\,(b-a)+c\,(c-a)\,(c-b)=9abc\ldots\text{(ii)},$$

which may be readily verified; for the left-hand side

$$=\Sigma a\,\{a^2-a\,(b+c)+bc\}$$
$$=\Sigma a\,\{2a^2+bc\},\ \text{since } b+c=-a,\ \&c.$$
$$=2\,(a^3+b^3+c^3)+3abc$$
$$=9abc,\ \text{since } a^3+b^3+c^3=3abc.$$

§3. If the expressions forming the left-hand member of (i) be multiplied together as they stand, the product

$$= 3 + \frac{\beta c}{\gamma b} + \frac{\gamma b}{\beta c} + \frac{\gamma a}{\alpha c} + \frac{\alpha c}{\gamma a} + \frac{\alpha b}{\beta a} + \frac{\beta a}{\alpha b}.$$

where for the moment α, β, γ denote $b-c$, $c-a$, $a-b$, so that (i) becomes

$$\frac{\beta^2 c^2 + \gamma^2 b^2}{\beta\gamma bc} + \frac{\gamma^2 a^2 + \alpha^2 c^2}{\gamma\alpha ca} + \frac{\alpha^2 b^2 + \beta^2 a^2}{\alpha\beta ab} = 6,$$

that is

$$\frac{(\beta c - \gamma b)^2}{\beta\gamma bc} + \frac{(\gamma a - \alpha c)^2}{\gamma\alpha ca} + \frac{(\alpha b - \beta a)^2}{\alpha\beta ab} = 0.$$

Now
$$\begin{aligned}\beta c - \gamma b &= c(c-a) - b(a-b)\\ &= c^2 + b^2 - a(b+c)\\ &= a^2 + b^2 + c^2,\end{aligned}$$

so that the formula becomes

$$\Sigma \frac{(a^2+b^2+c^2)^2}{bc(c-a)(a-b)} = 0,$$

that is
$$\Sigma \frac{(a^2+b^2+c^2)^2}{(c^2-a^2)(a^2-b^2)} = 0,$$

that is
$$(b^2-c^2) + (c^2-a^2) + (a^2-b^2) = 0.$$

§4. I now proceed to consider the value of the right-hand side of (i) when a, b, c are not subject to the condition $a+b+c=0$.

Proceeding as in §2, the left-hand side reduces as before to

$$\frac{a(a-b)(a-c) + b(b-c)(b-a) + c(c-a)(c-b)}{abc},$$

and it is easy to see that, identically,

$$\begin{aligned}&a(a-b)(a-c) + b(b-c)(b-a) + c(c-a)(c-b)\\ &\quad = 9abc + (a+b+c)(a^2+b^2+c^2-2bc-2ca-2ab) \ldots \text{(iii)},\end{aligned}$$

so that when a, b, c are unrestricted

$$\begin{aligned}&\left(\frac{b-c}{a} + \frac{c-a}{b} + \frac{a-b}{c}\right)\left(\frac{a}{b-c} + \frac{b}{c-a} + \frac{c}{a-b}\right)\\ &\quad = 9 + \frac{(a+b+c)(a^2+b^2+c^2-2bc-2ca-2ab)}{abc} \ldots\ldots \text{(iv.)}.\end{aligned}$$

The second member of this identity reduces to its first term both when $a+b+c=0$ and when $\sqrt{a}+\sqrt{b}+\sqrt{c}=0$; putting a^2, b^2, c^2 for a, b, c (iv) may be written

$$\left(\frac{b^2-c^2}{a^2}+\frac{c^2-a^2}{b^2}+\frac{a^2-b^2}{c^2}\right)\left(\frac{a^2}{b^2-c^2}+\frac{b^2}{c^2-a^2}+\frac{c^2}{a^2-b^2}\right)=$$

$$9-\frac{(a^2+b^2+c^2)(a+b+c)-(a+b+c)(a-b+c)(a+b-c)}{a^2b^2c^2} \quad \ldots(\mathrm{v}),$$

so that, if $a+b+c=0$,

$$\left(\frac{b-c}{a}+\frac{c-a}{b}+\frac{a-b}{c}\right)\left(\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}\right)=9,$$

and also

$$\left(\frac{b^2-c^2}{a^2}+\frac{c^2-a^2}{b^2}+\frac{a^2-b^2}{c^2}\right)\left(\frac{a^2}{b^2-c^2}+\frac{b^2}{c^2-a^2}+\frac{c^2}{a^2-b^2}\right)=9.$$

Each of these equations is however readily deducible from the other, for when $a+b+c=0$,

$$\frac{b-c}{a}=-\frac{(b+c)(b-c)}{a^2}=-\frac{b^2-c^2}{a^2}, \text{ \&c.}$$

§ 5. If a, b, c be the sides of a triangle, and A, B, C the opposite angles, then the second term on the right-hand side of (v)

$$=16S^2\frac{a^2+b^2+c^2}{a^2b^2c^2},$$

S denoting the area of the triangle,

$$=\frac{a^2+b^2+c^2}{R^2}=4(\sin^2A+\sin^2B+\sin^2C),$$

where R denotes the radius of the circumscribed circle.

Thus from (v) we obtain the trigonometrical formula

$$\left\{\frac{\sin(B-C)}{\sin A}+\frac{\sin(C-A)}{\sin B}+\frac{\sin(A-B)}{\sin C}\right\}$$

$$\times\left\{\frac{\sin A}{\sin(B-C)}+\frac{\sin B}{\sin(C-A)}+\frac{\sin C}{\sin(A-B)}\right\}$$

$$=9-4(\sin^2A+\sin^2B+\sin^2C)\ldots(\mathrm{vi}).$$

From this we may also deduce the formula

$$\frac{\sin(A-B)\sin(A-C)}{\sin B\sin C}+\frac{\sin(B-C)\sin(B-A)}{\sin C\sin A}$$

$$+\frac{\sin(C-A)\sin(C-B)}{\sin A\sin B}=9-4(\sin^2A+\sin^2B+\sin^2C)\ldots(\text{vii}),$$

which corresponds to (iii).

It is easy to verify (vii), for

$$\sin A\sin(A-B)\sin(A-C)=\tfrac{1}{4}(\sin 2B+\sin 2C-\sin 2A+\sin 4A);$$

and therefore

$$\sin A\sin(A-B)\sin(A-C)+\sin B\sin(B-A)\sin(B-C)$$

$$+\sin C\sin(C-A)\sin(C-B)$$

$$=\tfrac{1}{4}(\sin 2A+\sin 2B+\sin 2C+\sin 4A+\sin 4B+\sin 4C)$$

$$=\sin A\sin B\sin C-\sin 2A\sin 2B\sin 2C$$

$$=\sin A\sin B\sin C(1-8\cos A\cos B\cos C)$$

$$=\sin A\sin B\sin C(9-8-8\cos A\cos B\cos C)$$

$$=\sin A\sin B\sin C(9-4\sin^2A-4\sin^2B-4\sin^2C).$$

§ 6. The formula (i), in which $a+b+c=0$, may also be written in the form

$$\left(\frac{b-c}{b+c}+\frac{c-a}{c+a}+\frac{a-b}{a+b}\right)\left(\frac{b+c}{b-c}+\frac{c+a}{c-a}+\frac{a+b}{a-b}\right)=9\ldots(\text{viii}),$$

and it seems a matter of interest to determine the value of the right-hand side, where a, b, c are not subject to the condition $a+b+c=0$.

Supposing a, b, c to be unrestricted, the left-hand side of (viii)

$$=\frac{(b-c)(c+a)(a+b)+(c-a)(a+b)(b+c)+(a-b)(b+c)(c+a)}{(b+c)(c+a)(a+b)}$$

$$\times\frac{(b+c)(c-a)(a-b)+(c+a)(a-b)(b-c)+(a+b)(b-c)(c-a)}{(b-c)(c-a)(a-b)}$$

which

$$=\frac{(b+c)(a-b)(a-c)+(c+a)(b-c)(b-a)+(a+b)(c-a)(c-b)}{(b+c)(c+a)(a+b)}\ldots(\alpha),$$

for

$$(b-c)(c+a)(a+b)+(c-a)(a+b)(b+c)+(a-b)(b+c)(c+a)$$

$$=-(b-c)(c-a)(a-b)\ldots(\text{ix}).$$

[This last result may of course be proved at once by direct multiplication; but it may be simply deduced from the well-known identity

$$bc(b-c)+ca(c-a)+ab(a-b)=-(b-c)(c-a)(a-b)$$

as follows. The right-hand member of the equation last written is clearly unaffected by writing $a+\delta$, $b+\delta$, $c+\delta$ for a, b, c (δ being any arbitrary quantity), while the left-hand member becomes

$$(b+\delta)(c+\delta)(b-c)+(c+\delta)(a+\delta)(c-a)+(a+\delta)(b+\delta)(a-b).$$

Put now $\delta=-a-b-c$ and this expression becomes

$$(c+a)(a+b)(b-c)+(a+b)(b+c)(c-a)+(b+c)(c+a)(a-b)].$$

The numerator of (α), on multiplying out the terms, becomes

$$b^2c+bc^2+c^2a+ca^2+a^2b+ab^2-6abc,$$

which $=(b+c)(c+a)(a+b)-8abc$

$$=9(b+c)(c+a)(a+b)-8(a+b+c)(bc+ca+ab),$$

so that, identically,

$$\left(\frac{b-c}{b+c}+\frac{c-a}{c+a}+\frac{a-b}{a+b}\right)\left(\frac{b+c}{b-c}+\frac{c+a}{c-a}+\frac{a+b}{a-b}\right)$$
$$=9-8\frac{(a+b+c)(bc+ca+ab)}{(b+c)(c+a)(a+b)} \quad \ldots(\text{x}).$$

The second term of the right-hand member therefore vanishes when $a+b+c=0$, and also when $bc+ca+ab=0$; that is, when

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0.$$

These are in effect equivalent to only one condition, as the left-hand member is clearly unaltered by the substitution of $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ for a, b, c.

§7. If the expressions forming the left-hand member of (x) be multiplied together as they stand, as in §3, we obtain the identity

$$\frac{b-c}{b+c}\frac{c+a}{c-a}+\frac{b+c}{b-c}\frac{c-a}{c+a}+\frac{c-a}{c+a}\frac{a+b}{a-b}$$
$$+\frac{c+a}{c-a}\frac{a-b}{a+b}+\frac{a-b}{a+b}\frac{b+c}{b-c}+\frac{a+b}{a-b}\frac{b-c}{b+c}$$
$$=6-8\frac{(a+b+c)(bc+ca+ab)}{(b+c)(c+a)(a+b)} \quad \ldots\ldots\ldots\ldots\ldots (\text{xi}).$$

This identity may be readily verified, for the left-hand side

$$= \Sigma \frac{(b-c)^2(c+a)^2 + (b+c)^2(c-a)^2}{(b^2-c^2)(c^2-a^2)}$$

$$= 2\Sigma \frac{b^2c^2 + c^2a^2 + a^2b^2 + c^4 - 4abc^2}{(b^2-c^2)(c^2-a^2)},$$

whence the identity reduces to

$$a^4(b^2-c^2) + b^4(c^2-a^2) + c^4(a^2-b^2)$$
$$- 4a^2bc(b^2-c^2) - 4b^2ca(c^2-a^2) - 4c^2ab(a^2-b^2)$$
$$= 3(b^2-c^2)(c^2-a^2)(a^2-b^2)$$
$$- 4(a+b+c)(bc+ca+ab)(b-c)(c-a)(a-b),$$

which is at once seen to be true, since

$$a^4(b^2-c^2) + b^4(c^2-a^2) + c^4(a^2-b^2) = -(b^2-c^2)(c^2-a^2)(a^2-b^2)$$

and

$$a(b^2-c^2) + b(c^2-a^2) + c(a^2-b^2) = (b-c)(c-a)(a-b).$$

§ 8. Supposing, as in § 5, that a, b, c are the sides of a triangle, (x) gives rise to the theorem

$$\{\tan\tfrac{1}{2}A\tan\tfrac{1}{2}(B-C) + \tan\tfrac{1}{2}B\tan\tfrac{1}{2}(C-A) + \tan\tfrac{1}{2}C\tan\tfrac{1}{2}(A-B)\}$$
$$\times \{\cot\tfrac{1}{2}A\cot\tfrac{1}{2}(B-C) + \cot\tfrac{1}{2}B\cot\tfrac{1}{2}(C-A) + \cot\tfrac{1}{2}C\cot\tfrac{1}{2}(A-B)\}$$
$$= 9 - 8\frac{(a+b+c)(bc+ca+ab)}{(b+c)(c+a)(a+b)},$$

and from (xi) we have

$$\frac{\tan\frac{1}{2}A\tan\frac{1}{2}(B-C)}{\tan\frac{1}{2}B\tan\frac{1}{2}(C-A)} + \frac{\tan\frac{1}{2}B\tan\frac{1}{2}(C-A)}{\tan\frac{1}{2}A\tan\frac{1}{2}(B-C)} + \frac{\tan\frac{1}{2}B\tan\frac{1}{2}(C-A}{\tan\frac{1}{2}C\tan\frac{1}{2}(A-B)}$$
$$+ \frac{\tan\frac{1}{2}C\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}B\tan\frac{1}{2}(C-A)} + \frac{\tan\frac{1}{2}C\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}A\tan\frac{1}{2}(B-C)} + \frac{\tan\frac{1}{2}A\tan\frac{1}{2}(B-C)}{\tan\frac{1}{2}C\tan\frac{1}{2}(A-B)}$$
$$= 6 - 8\frac{(a+b+c)(bc+ca+ab)}{(b+c)(c+a)(a+b)} \quad \ldots\ldots\ldots \text{(xii)}.$$

§ 9. It may be mentioned that the theorem given in vol. VIII. p. 56, and referred to on p. 32 of the present volume, viz. that if $a+b+c=0$, then

$$a(b-c)^3 + b(c-a)^3 + c(a-b)^3 = 0 \quad \ldots\ldots\ldots \text{(xiii)},$$

leads to the identity

$$a(b-c)^3 + b(c-a)^3 + c(a-b)^3 = (a+b+c)(b-c)(c-a)(a-b)\ldots(\text{xiv}),$$

in which a, b, c are unrestricted.

It follows from (xiv) that if $a+b+c=0$,

$$(b^3-c^3)(b-c)^2 + (c^3-a^3)(c-a)^2 + (a^3-b^3)(a-b)^2 = 0\ldots(\text{xv}),$$

and, supposing a, b, c unrestricted, we have

$$(b^3-c^3)(b-c)^2 + (c^3-a^3)(c-a)^2 + (a^3-b^3)(a-b)^2 = 2(a+b+c)(b-c)(c-a)(a-b)\ldots(\text{xvi}).$$

June 29, 1880.

MATHEMATICAL NOTES.

On some algebraical expressions which are unaltered by certain substitutions.

§ 1. As is well known, the determinant

$$\begin{vmatrix} a_1, & a_2, & a_3, & \ldots a_n \\ a_n, & a_1, & a_2, & \ldots a_{n-1} \\ a_{n-1}, & a_n, & a_1, & \ldots a_{n-2} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_2, & a_3, & a_4, & \ldots a_1 \end{vmatrix} \ldots\ldots\ldots\ldots\ldots\ldots (\text{i})$$

$$= (a_1 + a_2 \ldots + a_n) \Pi (a_1 + \omega a_2 \ldots + \omega^{n-1} a_n),$$

where in the $n-1$ factors to which the product symbol Π refers ω denotes successively the n^{th} roots of unity, unity itself excepted.

Now if all the quantities $a_1, a_2, \ldots a_n$ be increased by the same quantity h, $a_1 + \omega a_2 \ldots + \omega^{n-1} a_n$ remains unaltered, since

$$(1 + \omega \ldots + \omega^{n-1})\, h = 0,$$

so that, if $a_1, a_2, \ldots a_n$ be replaced respectively by $a_1 + h$, $a_2 + h$, $\ldots a_n + h$, the determinant (i) is altered in the ratio of $a_1 + a_2 \ldots + a_n$ to $a_1 + a_2 \ldots + a_n + nh$.

Dividing out by $a_1 + a_2 \ldots + a_n$, the only effect of which is to replace each constituent in the first column by unity,

we have

$$\begin{vmatrix} 1, & a_2, & a_3, & \ldots a_n \\ 1, & a_1, & a_2, & \ldots a_{n-1} \\ 1, & a_n, & a_1, & \ldots a_{n-2} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ 1, & a_2, & a_4, & \ldots a_1 \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

$$= \Pi\,(a_1 + \omega a_2 \ldots + \omega^{n-1} a_n),$$

so that the determinant (ii) is unaltered by the substitution of $a_1 + h$, $a_2 + h$, $\ldots a_n + h$ for a_1, a_2, $\ldots a_n$. From this result the two following theorems may be deduced, the first of which is obtained by putting

$$h = -(a_1 + a_2 \ldots + a_n),$$

and the second by putting

$$h = -\tfrac{1}{2}(a_1 + a_2 \ldots + a_n).$$

I. If a_1, a_2, $\ldots a_n$ be replaced by $a_2 + a_3 \ldots + a_n$, $a_1 + a_3 \ldots + a_n$, $\ldots a_1 + a_2 \ldots + a_{n-1}$, *i.e.* if each of the n quantities be replaced by the sum of all the others, the value of the determinant (i) is changed in the ratio of 1 to $(-)^{n-1}(n-1)$, while the value of the determinant (ii) remains unaltered in magnitude, but undergoes a change of sign if n be even.

II. If a_1, a_2, $\ldots a_n$ be replaced by $\frac{1}{2}(-a_1 + a_2 \ldots + a_n)$, $\frac{1}{2}(a_1 - a_2 \ldots + a_n)$, $\ldots \frac{1}{2}(a_1 + a_2 \ldots - a_n)$, the value of the determinant (i) is changed in the ratio of 1 to $(-)^{n-1}\frac{1}{2}(n-2)$, while the value of (ii) remains unaltered in magnitude, but undergoes a change of sign if n be even.

If instead of being replaced by $\frac{1}{2}(-a_1 + a_2 \ldots + a_n)$, ... the original quantities be replaced by $-a_1 + a_2 \ldots + a_n$, $a_1 - a_2 \ldots + a_n$, ... the values of the determinants are altered in the ratios of 1 to $(-)^{n-1}2^{n-1}(n-2)$ and 1 to $(-)^{n-1}2^{n-1}$ respectively.

§ 2. As an example, consider the determinants of three rows, viz.

$$\begin{vmatrix} a, & b, & c \\ c, & a, & b \\ b, & c, & a \end{vmatrix},$$

which $= a^3 + b^3 + c^3 - 3abc$;

and $$\begin{vmatrix} 1, & b, & c \\ 1, & a, & b \\ 1, & c, & a \end{vmatrix},$$

which $$= a^3 + b^3 + c^3 - bc - ca - ab.$$

The theorem I. shows that

$$(b+c)^3 + (c+a)^3 + (a+b)^3 - 3(b+c)(c+a)(a+b) = 2(a^3 + b^3 + c^3 - 3abc),$$

$$(b+c)^2 + (c+a)^2 + (a+b)^2 - (c+a)(a+b) - (a+b)(b+c) - (b+c)(c+a) = a^2 + b^2 + c^2 - bc - ca - ab;$$

and II. shows that

$$(b+c-a)^3 + (c+a-b)^3 + (a+b-c)^3 - 3(b+c-a)(c+a-b)(a+b-c) = 4(a^3 + b^3 + c^3 - 3abc),$$

$$(b+c-a)^2 + (c+a-b)^2 + (a+b-c)^2 - (c+a-b)(a+b-c) - (a+b-c)(b+c-a) - (b+c-a)(c+a-b) = 4(a^2 + b^2 + c^2 - bc - ca - ab).$$*

§ 3. When the determinant (i) is of an even order, it is also divisible by a second real factor, viz. $a_1 - a_2 + a_3 \ldots - a_n$, and, since this factor is unaltered by increasing all the quantities $a_1, \ldots a_n$ by h, it is evident that the two theorems I. and II., so far as they relate to the determinant (ii), hold good also with regard to the expression of the $(n-2)^{\text{th}}$ order, which is left when the real factors $a_1 + a_2 \ldots + a_n$ and $a_1 - a_2 \ldots - a_n$ are thrown out, except that the expression undergoes change of sign when n is uneven, instead of when n is even.

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* If in the last two identities a, b, c be replaced by a^2, b^2, c^2, where a, b, c are the sides of a triangle, so that $b^2 + c^2 - a^2 = 2bc \cos A$, &c. we obtain the trigonometrical theorems: If $A + B + C = \pi$, then

$$\sin^2 B \sin^2 C \cos^2 A + \sin^2 C \sin^2 A \cos^2 B + \sin^2 A \sin^2 B \cos^2 C - 3 \sin^2 A \sin^2 B \sin^2 C \cos A \cos B \cos C = \tfrac{1}{2}(\sin^6 A + \sin^6 B + \sin^6 C - 3 \sin^2 A \sin^2 B \sin^2 C),$$

and

$$\sin^2 B \sin^2 C \cos^2 A + \sin^2 C \sin^2 A \cos^2 B + \sin^2 A \sin^2 B \cos^2 C - \sin^2 A \sin B \sin C \cos B \cos C - \sin^2 B \sin C \sin A \cos C \cos A - \sin^2 C \sin A \sin B \cos A \cos B = \sin^2 A + \sin^2 B + \sin^2 C - \sin B \sin C - \sin C \sin A - \sin A \sin B.$$

For example,

$$\begin{vmatrix} a, & b, & c, & d \\ d, & a, & b, & c \\ c, & d, & a, & b \\ b, & c, & d, & a \end{vmatrix} = a^4 - b^4 + c^4 - d^4 - 2a^2c^2 + 2b^2d^2 - 4a^2bd + 4b^2ac - 4c^2bd + 4d^2ac,$$

and, throwing out the factors $a+b+c+d$ and $a-b+c-d$, the remaining expression is

$$a^2 + b^2 + c^2 + d^2 - 2ac - 2bd,$$

which is unaltered by the substitution of $b+c+d$, $a+c+d$, $a+b+d$, $a+b+c$ for a, b, c, d, and also by that of $\frac{1}{2}(-a+b+c+d)$, $\frac{1}{2}(a-b+c+d)$, $\frac{1}{2}(a+b-c+d)$, $\frac{1}{2}(a+b+c-d)$ for a, b, c, d. With respect to this last result it may be remarked that the two portions $a^2+b^2+c^2+d^2$ and $2ac+2bd$, of which the expression is composed, each remain unaltered by this substitution.

The expanded values of the determinants of the form (i) having five and six rows are given in the *Quarterly Journal* vol. XV. p. 354 and vol. XVI. p. 33; the former contains 26 terms and the latter 68 terms. The expanded value of the determinant of the form (ii) having five rows is given in vol. XV. p. 350; it contains 70 terms.

J. W. L. GLAISHER.

Addition to a note "a trigonometrical identity" (vol IX. p. 191).

From the form of the identity (1) we see that there must be a third angle whose sine and cosine (s_2, c_2) satisfy the equations

$$\left.\begin{aligned} c_1^{\frac{4}{3}} s^{\frac{4}{3}} - c^{\frac{4}{3}} s_1^{\frac{4}{3}} &= c_2^{\frac{4}{3}} (s^{\frac{4}{3}} c^{\frac{4}{3}} - s_1^{\frac{4}{3}} c_1^{\frac{4}{3}}) \\ s_1^{\frac{4}{3}} c^{\frac{4}{3}} - s^{\frac{4}{3}} c_1^{\frac{4}{3}} &= s_2^{\frac{4}{3}} (s^{\frac{4}{3}} c^{\frac{4}{3}} - s_1^{\frac{4}{3}} c_1^{\frac{4}{3}}) \end{aligned}\right\} \text{.....(i);}$$

whence, multiplying these equations by $c^{\frac{4}{3}} c_1^{\frac{4}{3}}$ and $s^{\frac{4}{3}} s_1^{\frac{4}{3}}$ respectively, we have

$$\{cc_1c_2\}^{\frac{4}{3}} + \{ss_1s_2\}^{\frac{4}{3}} = 1 \text{ .....(2)}$$

and hence we might infer from the symmetry of the equation, that

$$\left.\begin{aligned} c_2^{\frac{4}{3}} s_1^{\frac{4}{3}} - c_1^{\frac{4}{3}} s_2^{\frac{4}{3}} &= c^{\frac{4}{3}} (s_1^{\frac{4}{3}} c_1^{\frac{4}{3}} - s_2^{\frac{4}{3}} c_2^{\frac{4}{3}}) \\ s_2^{\frac{4}{3}} c_1^{\frac{4}{3}} - s_1^{\frac{4}{3}} c_2^{\frac{4}{3}} &= s^{\frac{4}{3}} (s_1^{\frac{4}{3}} c_1^{\frac{4}{3}} - s_2^{\frac{4}{3}} c_2^{\frac{4}{3}}) \end{aligned}\right\} \text{.....(ii),}$$

$$\left.\begin{aligned} c^{\frac{1}{3}}s_2^{\frac{1}{3}} - c_2^{\frac{1}{3}}s^{\frac{1}{3}} &= c_1^{\frac{1}{3}}(s_2^{\frac{1}{3}}c_2^{\frac{1}{3}} - s^{\frac{1}{3}}c^{\frac{1}{3}}) \\ s^{\frac{1}{3}}c_2^{\frac{1}{3}} - s_2^{\frac{1}{3}}c^{\frac{1}{3}} &= s_1^{\frac{1}{3}}(s_2^{\frac{1}{3}}c_2^{\frac{1}{3}} - s^{\frac{1}{3}}c^{\frac{1}{3}}) \end{aligned}\right\} \quad \text{.....(iii).}$$

It follows that any one of the six equations (i), (ii), (iii) being true, the other five are so also. Further, c_1, s_1, c_2, s_2, being supposed known quantities, equations (ii) give the roots of (2) considered as a cubic equation in c^2 or $s^2 (= 1 - c^2)$.

HARRY HART.

On the path of a projectile.

Let P be any point in the path of a projectile whose highest point is A. Let the horizontal through P, and the normal at P meet the vertical through A in N and G respectively. Then, if t be the time of passing from P to A,

vertical velocity at $P = gt$,

horizontal ..... $=$ const. $= c$,

and $$PN = ct.$$

Then, since the sides of triangle PNG are perpendicular to the velocity at P and its horizontal and vertical components,

$$\frac{NG}{PN} = \frac{c}{gt},$$

or $$NG = \frac{c^2}{g};$$

that is, the subnormal is constant, or the path is a parabola whose latus rectum = twice subnormal $= \frac{2c^2}{g}$.

If v be the velocity at P

$$\frac{v}{c} = \frac{PG}{NG}; \text{ i.e. } v = \frac{g}{c}PG,$$

or $$v^2 = \frac{g^2}{c^2}PG^2 = 2g\,SP,$$

S being the focus.

HARRY HART.

An elliptic function identity.

The following elliptic function identity is suggested by the trigonometrical identity noted on p. 124, vol IX. (December, 1879).

$$
\begin{aligned}
&[\operatorname{sn}\gamma \operatorname{sn}(\alpha-\beta)+\operatorname{sn}\alpha \operatorname{sn}(\beta-\gamma)+\operatorname{sn}\beta \operatorname{sn}(\gamma-\alpha)]\\
&\quad\times[\operatorname{sn}\alpha \operatorname{sn}\beta \operatorname{sn}\gamma+\operatorname{sn}\alpha \operatorname{sn}(\alpha-\beta)\operatorname{sn}(\gamma-\alpha)\\
&\quad+\operatorname{sn}\beta \operatorname{sn}(\beta-\gamma)\operatorname{sn}(\alpha-\beta)+\operatorname{sn}\gamma \operatorname{sn}(\gamma-\alpha)\operatorname{sn}(\beta-\gamma)]\\
&\quad=-\operatorname{sn}\beta \operatorname{sn}\gamma \operatorname{sn}(\beta-\gamma)[\operatorname{sn}^2\beta+\operatorname{sn}^2\gamma+\operatorname{sn}^2(\beta-\gamma)]\\
&\qquad-\operatorname{sn}\gamma \operatorname{sn}\alpha \operatorname{sn}(\gamma-\alpha)[\operatorname{sn}^2\gamma+\operatorname{sn}^2\alpha+\operatorname{sn}^2(\gamma-\alpha)]\\
&\qquad-\operatorname{sn}\alpha \operatorname{sn}\beta \operatorname{sn}(\alpha-\beta)[\operatorname{sn}^2\alpha+\operatorname{sn}^2\beta+\operatorname{sn}^2(\alpha-\beta)]\\
&\qquad-\operatorname{sn}(\beta-\gamma)\operatorname{sn}(\gamma-\alpha)\operatorname{sn}(\alpha-\beta)\\
&\qquad\times[\operatorname{sn}^2(\beta-\gamma)+\operatorname{sn}^2(\gamma-\alpha)+\operatorname{sn}^2(\alpha-\beta)].
\end{aligned}
$$

M. M. U. WILKINSON.

TRANSACTIONS OF SOCIETIES.

London Mathematical Society.

Thursday, April 8th, 1880., C. W. Merrifield, Esq., F.R.S., *President*, in the Chair. Mr. J. Barnard was elected a member, and Mr. T. Olver Harding was admitted into the Society. The following communications were made to the Society: "A form of the equations determining the foci and directrices of a conic whose equation in Cartesian coordinates is given," Prof. Wolstenholme; "The application of elliptic coordinates and Lagrange's equations of motion to Euler's problem of two centres of force," Mr. A. G. Greenhill; "Theorems in the calculus of operations," Mr. J. J. Walker; "On the equilibrium of cords and beams in certain cases," Mr. W. J. C. Sharp; "On steady motion and vortex motion in an incompressible viscous fluid," Mr. T. Craig; "On functions analogous to Laplace's functions," Mr. E. J. Routh, F.R.S.

Thursday, May 13th, 1880, C. W. Merrifield, Esq., F.R.S., *President*. in the Chair. The following communications were made to the Society: "On Cremonian congruences," Dr. Hirst, F.R.S.; "On some statical and kinematical theorems," Prof. Minchin, M.A.; "On a class of analytical problems," Prof. Cayley, F.R.S.

Thursday, June 10th, 1880, C. W. Merrifield, Esq., F.R.S., *President*, in the Chair. The following communications were made to the Society: "On a binomial biordinal and the arbitrary constants of its complete solution," Sir James Cockle, F.R.S.; "On the focal conics of a bicircular quantic," H. Hart, M.A.; "Preliminary note on a generalization of Pfaff's problem," H. W. Lloyd Tanner, M.A.; "On the resultant of a cubic and a quadric binary form," Prof. Cayley, F.R.S.; "On the theory of the focal distances of points on plane curves," W. J. C. Sharp, M.A.; "Note on the equation of the two planes which can be drawn through two given points to touch a quadric," H. M. Taylor, M.A.

R. TUCKER, M.A., *Hon. Sec.*

NOTES ON QUATERNIONS.

By *Edwyn Anthony, M.A.*

THE following are desultory notes on some elementary quaternion theorems.

I. $S\alpha_1 \ldots \alpha_n =$ any arrangement cyclically the same...(1),

$$S\alpha_1 \ldots \alpha_n = (-1)^n S\alpha_n \ldots\ldots \alpha_1 \ldots\ldots\ldots\

II. If $\alpha_1 \ldots \alpha_n$ are coplanar vectors, then, *n odd*, $S\alpha_1 \ldots \alpha_n = 0$ always; $V\alpha_1 \ldots \alpha_n = 0$ never, unless one of the vectors vanish; *n even*, $S\alpha_1 \ldots \alpha_n = 0$, only when the sum of the angles between each pair of vectors taken in order is some odd multiple of a right angle, and $V\alpha_1 \ldots \alpha_n = 0$, only when the said sum is some even multiple of a right angle.

Also $\alpha_1 \ldots \alpha_n = S\alpha_1\alpha_2.\alpha_3 \ldots \alpha_n + \beta\alpha_3.\alpha_4 \ldots \alpha_n$ (where β is a vector perpendicular to the plane of the vectors) $= S\alpha_1\alpha_2.\alpha_3 \ldots \alpha_n + \gamma\alpha_4 \ldots \alpha_n$ (where γ is a coplanar vector); therefore $S\alpha_1 \ldots \alpha_n = 0$ holds for the value n, if true for the value $(n-2)$. But it is true when $n = 1$, and hence for all odd values of n.

When n is even let a_1, a_2, &c. be the angles between the pairs of vectors, then

$$\alpha_1 \ldots \alpha_n = (\cos a_1 + \sin a_1 . \varepsilon)(\cos a_2 + \sin a_2 . \varepsilon) \ldots$$
$$= \cos(a_1 + a_2 + \ldots) \pm \sin(a_1 + a_2 + \ldots).\varepsilon;$$

therefore $$S\alpha_1 \ldots \alpha_n = \cos(a_1 + a_2 + \ldots),$$

and $$V\alpha_1 \ldots \alpha_n = \pm \sin(a_1 + a_2 + \ldots).\varepsilon.$$

When n is odd it is plain from the above, since

$$V\alpha_1 \ldots \alpha_n = S\alpha_2 \ldots \alpha_n . \alpha_1 + V(\alpha_1 V\alpha_2 \ldots \alpha_n),$$

that $V\alpha_1 \ldots \alpha_n$ cannot vanish unless one of the vectors, vanishes.

III. The theorem (Tait's *Quaternions*, 2nd ed., p. 53) that if α, β, γ be the sides of a triangle taken in order, the tangents to the circumscribing circle at the angles of the triangle are parallel respectively to $\alpha\beta\gamma$, $\beta\gamma\alpha$, $\gamma\alpha\beta$ may be extended to any odd-sided polygon inscribable in a circle. For if $ABCDEF$ be a hexagon inscribed in a circle whose sides in order, all but the last, are α, β, γ, δ, ε, then as is shown in Tait, AD is parallel to $\alpha\beta\gamma$. Similarly AF is parallel to $AD.DE.EF$, that is, to $\alpha\beta\gamma\delta\varepsilon$. Let F move up to A, then AF becomes the tangent at A, therefore the tangent at A is parallel to $\alpha\beta\gamma\delta\varepsilon$. And similarly the proposition may be proved for any odd-sided polygon.

Cor. It is obvious that if the polygon is even-sided the continued product of the vectors taken in order is a scalar.*

IV. Many of the general expressions in connection with the equation to a curve in the form $\rho = l\alpha + m\beta + n\gamma$, where α, β, γ are non-coplanar vectors, and l, m, n functions of

* The writer has since found the proposition as here stated in the *Lectures on Quaternions*, p. 309.

some scalar variable t, appear to be worthy of notice. We shall denote the first, second, &c. differential coefficients of l, m, n by l', m', n', l'', m'', n'', &c.

Plane Curves.

Equation to curve $\rho = l\alpha + m\beta$ .....(1).

Equation to tangent $\rho = l\alpha + m\beta + x(l'\alpha + m'\beta)$ .....(2).

If $X\alpha + Y\beta$ is parallel to the normal,

$$S(X\alpha + Y\beta)(l'\alpha + m'\beta) = 0,$$

whence, equation to normal is

$$\rho = l\alpha + m\beta + x\{\alpha(m'\beta^2 + l'S\alpha\beta) - \beta(l'\alpha^2 + m'S\alpha\beta)\}...(3).$$

Equating (3) to the normal at the consecutive point $l + dl$, &c., we get

$$x = \frac{(l'\alpha + m'\beta)^2}{V^2\alpha\beta(l'm'' - l''m')}.$$

Hence the equation to the evolute is

$$\rho = l\alpha + m\beta + \frac{(l'\alpha + m'\beta)^2}{V^2\alpha\beta(l'm'' - l''m')} \times \{\alpha(m'\beta^2 + l'S\alpha\beta) - \beta(l'\alpha^2 + m'S\alpha\beta)\}.....(4).$$

If α and β are perpendicular to one another the equation to the evolute reduces to

$$\rho = l\alpha + m\beta - \frac{l'^2\alpha^2 + m'^2\beta^2}{\alpha^2\beta^2(l'm''' - l''m')}(m'\beta^2\alpha - l'\alpha^2\beta),$$

and similarly for the normal. If also their tensors are equal, the equations to the normal and evolute become respectively

$$\rho = l\alpha + m\beta + x(m'\alpha - l'\beta),$$

$$\rho = l\alpha + m\beta - \frac{l'^2 + m'^2}{l'm'' - l''m'}(m'\alpha - l'\beta).$$

Examples. The hyperbola referred to its asymptotes,

$$\rho = t\alpha + \beta t^{-1},$$

$$\rho = ... + x(t\alpha - \beta t^{-1}),$$

$$\rho = ... + x\{\alpha(\cos\omega - t^{-2}) - \beta(1 - t^{-2}\cos\omega)\},$$

$$\rho = ... - \frac{t(t^2 + t^{-2} - 2\cos\omega)}{2\sin^2\omega}\{.....\},$$

where ω = angle between the asymptotes and

$$T\alpha = T\beta = \sqrt{(a^2 + b^2)}.$$

The hyperbola referred to conjugate diameters,

$$\rho = \sec\theta\alpha + \tan\theta\beta,$$

$$\rho = \ldots\ldots\ldots\ldots\ldots + x(\sin\theta\alpha + \beta),$$

$$\rho = \ldots\ldots\ldots\ldots\ldots + x \qquad (b^2\alpha - a^2\sin\theta\beta),$$

$$\rho = \ldots\ldots\ldots\ldots\ldots + \frac{\sec^2\theta\,(a^2\sin^2\theta + b^2)}{a^2b^2}\,(\ldots\ldots\ldots\ldots\ldots),$$

where α and β are conjugate semi-diameters. In the last two equations they are the semi-axes. The corresponding equations for the ellipse are of course very similar.

The curve whose Cartesian equation is $y^m = a^{m-n}x^n$,

$$\rho = t^m\alpha + t^n\beta,$$

$$\rho = \ldots\ldots\ldots + x(mt^m\alpha + nt^n\beta),$$

$$\rho = \ldots\ldots\ldots + x \qquad (nt^n\alpha - mt^m\beta),$$

$$\rho = \ldots\ldots\ldots + \frac{m^2t^{m-n} + n^2t^{m-n}}{mn(m-n)}\,(\ldots\ldots\ldots\ldots),$$

where $T\alpha = T\beta = a$. In the last two equations α and β must be perpendicular to one another.

The cycloid

$$\rho = (\theta - \sin\theta)\alpha + (1 - \cos\theta)\beta,$$

$$\rho = \ldots\ldots\ldots\ldots\ldots\ldots, + x\{(1 - \cos\theta)\alpha + \sin\theta\beta\},$$

$$\rho = \ldots\ldots\ldots\ldots\ldots\ldots + x\{\sin\theta\alpha - (1 - \cos\theta)\beta\},$$

$$\rho = (\theta + \sin\theta)\alpha - (1 - \cos\theta)\beta,$$

where the origin is at a double point, α and β are parallel and perpendicular respectively to the line on which the circle rolls, and $T\alpha = T\beta =$ radius of rolling circle.

Non-plane Curves. Taking α, β, γ a system of rectangular unit vectors, we have

$$S(l\alpha + m\beta + n\gamma)(l'\alpha + m'\beta + n'\gamma)(l''\alpha + m''\beta + n''\gamma),$$

$$= -\begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ l'', & m'', & n'' \end{vmatrix} = a_1 \text{ for shortness,}$$

and

$$V(l\alpha + m\beta + n\gamma)(l'\alpha + m'\beta + n'\gamma),$$

$$= \begin{vmatrix} \alpha, & \beta, & \gamma \\ l, & m, & n \\ l', & m', & n' \end{vmatrix}.$$

We shall denote by L, M, N the minors of the first determinant corresponding to l, m, n, and the first, second, &c., differential coefficients of L, M, N by $L', M', N', L'', M'', N''$, &c.

The equation to the curve being

$$\rho = l\alpha + m\beta + n\gamma \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5),$$

the equation to the tangent is

$$\rho = l\alpha + m\beta + n\gamma + x(l'\alpha + m'\beta + n'\gamma) \ldots\ldots\ldots (6).$$

The *normal plane* passes through $l\alpha + m\beta + n\gamma$, and is perpendicular to $l'\alpha + m'\beta + n'\gamma$. Its equation is therefore

$$S(l'\alpha + m'\beta + n'\gamma)\{\rho - (l\alpha + m\beta + n\gamma)\} = 0;$$

that is

$$S\rho(l'\alpha + m'\beta + n'\gamma) = -(ll' + mm' + nn') \ldots\ldots (7).$$

The *osculating plane* passes through two consecutive tangents. It is therefore perpendicular to

$$V(l'\alpha + m'\beta + n'\gamma)\{(l' + dl')\alpha + (m' + dm')\beta + (n' + dn')\gamma\},$$

that is to $\begin{vmatrix} \alpha, & \beta, & \gamma \\ l', & m', & n' \\ dl', & dm', & dn' \end{vmatrix}$.

Its equation is therefore

$$S\begin{vmatrix} \alpha, & \beta, & \gamma \\ l', & m', & n' \\ dl', & dm', & dn' \end{vmatrix}\{\rho - (l'\alpha + m'\beta + n'\gamma)\} = 0,$$

that is $S\rho(L\alpha + M\beta + N\gamma) = -(lL + Mm + nN) = a_1 \ldots (8).$

The *binormal* is perpendicular to the osculating plane. Hence its equation is

$$\rho = l\alpha + m\beta + n\gamma + (L\alpha + M\beta + N\gamma) \ldots\ldots\ldots (9).$$

The *principal normal* is perpendicular to the tangent and to the binormal. Hence, it is parallel to

$$V(l'\alpha + m'\beta + n'\gamma)(L\alpha + M\beta + N\gamma),$$

and its equation is

$$\rho = l\alpha + m\beta + n\gamma + x\begin{vmatrix} \alpha, & \beta, & \gamma \\ l', & m', & n' \\ L, & M, & N \end{vmatrix} \ldots\ldots\ldots (10).$$

At the common point (ρ) of three consecutive normal planes, we have

$$S\rho(l'\alpha + m'\beta + n'\gamma) = -(ll' + mm' + nn') = a \text{ (suppose)},$$

$$S\rho(l''\alpha + m''\beta + n''\gamma) = -(l'^2 + m'^2 + n'^2) - (ll'' + mm'' + nn'') = b \text{ (suppose)}$$

$$S\rho\,(l'''\alpha + m'''\beta + n'''\gamma) = -3(ll' + mm' + nn') - (ll''' + mm''' + nn''') = c \text{ (suppose).}$$

Therefore the vector to the centre of *spherical curvature*, ρ,

$$= \frac{\begin{vmatrix} 0, & \alpha, & \beta, & \gamma \\ a, & l', & m', & n' \\ b, & l'', & m'', & n'' \\ c, & l''', & m''', & n''' \end{vmatrix}}{\begin{vmatrix} l', & m', & n' \\ l'', & m'', & n'' \\ l''', & m''', & n''' \end{vmatrix}} \quad \text{.....(11).}$$

Of course, when we consider t variable, the equation represents the locus of the centres of spherical curvature. Similarly, since the centre of absolute curvature is the intersection of the osculating plane and of two consecutive normal planes, the vector to the *centre of absolute curvature*

$$= \frac{\begin{vmatrix} 0, & \alpha, & \beta, & \gamma \\ a_1, & L, & M, & N \\ a, & l', & m', & n' \\ b, & l'', & m'', & n'' \end{vmatrix}}{\begin{vmatrix} L, & M, & N \\ l', & m', & n' \\ l'', & m'', & n'' \end{vmatrix}} \quad \text{.....(12).}$$

The *angle of contingence* = the angle between two consecutive tangents

$$l'\alpha + m'\beta + n'\gamma,\ (l' + dl')\alpha + (m' + dm')\beta + (n' + dn')\gamma$$

= sine of the angle between these lines

$$= \frac{\{(l'^2 + m'^2 + n'^2)(l''^2 + m''^2 + n''^2) - (l'l'' + m'm'' + n'n'')^2\}^{\frac{1}{2}}}{l'^2 + m'^2 + n'^2} dt$$

$$= \frac{(L^2 + M^2 + N^2)^{\frac{1}{2}}}{l'^2 + m'^2 + n'^2} dt \quad \text{.....(13).}$$

And the *angle of torsion*, being the angle between consecutive osculating planes equals the above expression when L, M, N, L', M', N' are written for $l', m', n', l'', m'', n''$ respectively.

As an example, the equation to the helix is

$$\rho = a\cos\theta\alpha + a\sin\theta\beta + an\theta\gamma,$$

where a = radius of cylinder, and $n = \tan d$, d being the angle of the helix. The equation to the tangent is

$$\rho = a\cos\theta\alpha + a\sin\theta\beta + an\theta\gamma + x(\ \sin\theta\alpha - \ \cos\theta\beta - n\gamma),$$

binormal

$$\rho = \quad\ \ " \quad\ \ " \quad\ + x(n\sin\theta\alpha - n\cos\theta\beta + \ \gamma),$$

principal normal

$$\rho = \quad\ \ " \quad\ \ " \quad\ + x(\cos\theta\alpha + \sin\theta\beta),$$

$$S\rho(\sin\theta\alpha - \cos\theta\beta - n\gamma) = a^2n\theta, \text{ normal plane,}$$

$$S\rho(n\sin\theta\alpha - n\cos\theta + \gamma) = -an, \text{ osculating plane,}$$

$$\rho = -n^2a(\cos\theta\alpha + \sin\theta\beta) + na\gamma\theta,$$

vector to centre of spherical and of absolute curvature.

The rectifying line is parallel to γ.

Sine of semi-vertical angle of osculating cone $= \cos d$.

Angle of contingence $= \cos d.d\theta$.

Angle of torsion $= \sin d.d\theta$.

From the above most of the properties of the helix are at once apparent. Thus the tangent is inclined to the axis of the cylinder at a constant angle = the angle of the helix, and the equation to the locus of its intersection with any plane perpendicular to the axis is

$$\rho = a(\cos\theta + \theta\sin\theta)\alpha + a(\sin\theta - \cos\theta)\beta,$$

that is, the involute to the circle in which the plane cuts the cylinder. The principal normal intersects and is perpendicular to the axis. The osculating plane is therefore a normal plane to the cylinder on which the helix is traced, and the equation to the principal normal becomes, by considering x and θ variable, the equation to the helicoid. The radius of spherical and of absolute curvature each $= a\sec^2 d$; their centres coincide, and the locus of the common centre is a coaxial helix, which becomes a helix of the same angle, and traced on the same cylinder as the original helix when the angle of the latter $= 45°$. The axis of the osculating cone is parallel to the axis of the cylinder, and its semi-vertical angle = the complement of the angle of the helix. And the angle between consecutive radii of curvature, that is, between consecutive principal normals = the angle between the corresponding radii of a circular section of the cylinder.

A SYSTEM OF TRIGONOMETRICAL FORMULÆ.

By *J. W. L. Glaisher.*

§1. If the suffixes 1, 2, ...6 refer to the arguments A, $B-C$, ...$A-B$ according to the scheme

1	A
2	$B-C$
3	B
4	$C-A$
5	C
6	$A-B$

where A, B, C are any three quantities, and if s, c stand for sine and cosine, so that, for example, s_2 denotes $\sin B$, c_4 denotes $\cos(C-A)$, &c., then s_5, c_5, s_6, c_6 are connected with s_1, c_1, s_2, ...c_4 by the following system of formulæ:

$$s_5(s_1s_4+s_2s_3)+s_6(s_1s_3+s_2s_4)=0 \quad\ldots\ldots(1),$$
$$s_5(s_2c_3+s_4c_1)+s_6(s_1c_3+s_4c_2)=0 \quad\ldots\ldots(2),$$
$$c_5(s_1c_4+s_2c_3)+c_6(s_1c_3+s_2c_4)=0 \quad\ldots\ldots(3),$$
$$c_5(s_1s_4+c_2c_3)-c_6(s_2s_4+c_1c_3)=0 \quad\ldots\ldots(4),$$
$$s_5(s_1c_4+s_2c_3)-c_6(s_1s_3+s_2s_4)=0 \quad\ldots\ldots(5),$$
$$s_5(s_1s_4+c_2c_3)-c_6(s_1c_3+s_4c_2)=0 \quad\ldots\ldots(6),$$
$$s_6(s_1c_3+s_2c_4)+c_5(s_1s_4+s_2s_3)=0 \quad\ldots\ldots(7),$$
$$s_6(s_2s_4+c_1c_3)+c_5(s_2c_3+s_4c_1)=0 \quad\ldots\ldots(8),$$
$$s_5(s_1c_3+s_2c_4)-c_5(s_1s_3+s_2s_4)=0 \quad\ldots\ldots(9),$$
$$s_5(s_1s_2+c_3c_4)-c_5(s_1c_2+s_4c_3)=0 \quad\ldots\ldots(10),$$
$$s_5(s_2c_1+s_4c_3)-c_5(s_1s_2+s_3s_4)=0 \quad\ldots\ldots(11),$$
$$s_5(s_2s_4+c_1c_3)-c_5(s_1c_3+s_4c_2)=0 \quad\ldots\ldots(12),$$
$$s_6(s_1c_4+s_2c_3)+c_6(s_1s_4+s_2s_3)=0 \quad\ldots\ldots(13),$$
$$s_6(s_1s_2+c_3c_4)+c_6(s_2c_1+s_4c_3)=0 \quad\ldots\ldots(14),$$
$$s_6(s_1c_2+s_4c_3)+c_6(s_1s_2+s_3s_4)=0 \quad\ldots\ldots(15),$$
$$s_6(s_1s_4+c_2c_3)+c_6(s_2c_3+s_4c_1)=0 \quad\ldots\ldots(16),$$

$$s_5(s_1c_2+s_2c_1)-(s_1s_3+s_2s_4)=0 \quad \ldots\ldots\ldots(17),$$
$$s_5(s_1s_3+c_1c_3)-(s_1c_2+s_4c_3)=0 \quad \ldots\ldots\ldots(18),$$
$$s_5(s_2c_4+s_4c_2)-(s_1s_2+s_3s_4)=0 \quad \ldots\ldots\ldots(19),$$
$$s_5(s_3s_4+c_3c_4)-(s_1c_3+s_4c_3)=0 \quad \ldots\ldots\ldots(20),$$
$$s_6(s_1c_2+s_2c_1)+(s_1s_4+s_3s_3)=0 \quad \ldots\ldots\ldots(21),$$
$$s_6(s_2s_3+c_2c_3)+(s_2c_1+s_4c_3)=0 \quad \ldots\ldots\ldots(22),$$
$$s_6(s_1c_4+s_4c_1)+(s_1s_2+s_3s_4)=0 \quad \ldots\ldots\ldots(23),$$
$$s_6(s_3s_4+c_3c_4)+(s_2c_3+s_4c_1)=0 \quad \ldots\ldots\ldots(24),$$
$$c_5(s_1c_2+s_2c_1)-(s_1c_3+s_2c_4)=0 \quad \ldots\ldots\ldots(25),$$
$$c_5(s_2c_4+s_4c_2)-(s_2c_1+s_4c_3)=0 \quad \ldots\ldots\ldots(26),$$
$$c_5(s_1s_3+c_1c_3)-(s_1s_2+c_3c_4)=0 \quad \ldots\ldots\ldots(27),$$
$$c_5(s_3s_4+c_3c_4)-(s_2s_4+c_1c_3)=0 \quad \ldots\ldots\ldots(28),$$
$$c_6(s_1c_2+s_2c_1)-(s_1c_4+s_2c_3)=0 \quad \ldots\ldots\ldots(29),$$
$$c_6(s_1c_4+s_4c_1)-(s_1c_2+s_4c_2)=0 \quad \ldots\ldots\ldots(30),$$
$$c_6(s_2s_3+c_2c_3)-(s_1s_2+c_3c_4)=0 \quad \ldots\ldots\ldots(31),$$
$$c_6(s_3s_4+c_3c_4)-(s_1s_4+c_2c_3)=0 \quad \ldots\ldots\ldots(32).$$

§ 2. In these 32 formulæ there are involved 18 expressions containing the suffixes 1, 2, 3, 4 only. These expressions are

$$s_1s_2+s_3s_4, \quad s_1s_2+c_3c_4,$$
$$s_1s_3+s_2s_4, \quad s_1s_4+c_2c_3,$$
$$s_1s_4+s_2s_3, \quad s_2s_4+c_1c_3,$$
$$s_1c_2+s_4c_3, \quad s_1c_4+s_2c_3,$$
$$s_1c_3+s_2c_4, \quad s_2c_1+s_4c_3,$$
$$s_1c_3+s_4c_2, \quad s_2c_3+s_4c_1,$$
$$s_1s_3+c_1c_3, \quad s_1c_2+s_2c_1,$$
$$s_2s_3+c_2c_3, \quad s_1c_4+s_4c_1,$$
$$s_3s_4+c_3c_4, \quad s_2c_4+s_4c_2,$$

and each of them is equal to another expression, in which the sign of the second term is negative, viz. we have

$$s_1s_2+s_3s_4=c_1c_2-c_3c_4,$$
$$s_1s_3+s_2s_4=c_2c_4-c_1c_3,$$
$$s_1s_4+s_2s_3=c_1c_4-c_2c_3,$$
$$s_1s_2+c_3c_4=c_1c_2-s_3s_4,$$
$$s_1s_4+c_2c_3=c_1c_4-s_2s_3,$$

$$s_2s_4 + c_1c_3 = c_2c_4 - s_1s_3,$$
$$s_1c_2 + s_4c_3 = s_3c_4 - s_2c_1,$$
$$s_1c_3 + s_2c_4 = s_3c_1 - s_4c_2,$$
$$s_1c_3 + s_4c_2 = s_3c_1 - s_2c_4,$$
$$s_1c_4 + s_2c_3 = s_3c_2 - s_4c_1,$$
$$s_2c_1 + s_4c_3 = s_3c_4 - s_1c_2,$$
$$s_2c_3 + s_4c_1 = s_3c_2 - s_1c_4,$$
$$s_1s_3 + c_1c_3 = c_2c_4 - s_2s_4,$$
$$s_2s_3 + c_2c_3 = c_1c_4 - s_1s_4,$$
$$s_3s_4 + c_3c_4 = c_1c_2 - s_1s_2,$$
$$s_1c_2 + s_2c_1 = s_3c_4 - s_4c_3,$$
$$s_1c_4 + s_4c_1 = s_3c_2 - s_2c_3,$$
$$s_2c_4 + s_4c_2 = s_3c_1 - s_1c_3.$$

§ 3. By substituting in the formulæ (1), ...(32) for any of these expressions their equivalents with the negative sign, other formulæ of the same class may be deduced. The system obtained by substituting for every one of the expressions its equivalent is as follows:

$$s_5(c_1c_4 - c_2c_3) + s_6(c_2c_4 - c_1c_3) = 0 \text{.....(i)},$$
$$s_5(s_1c_4 - s_3c_2) + s_6(s_2c_4 - s_3c_1) = 0 \text{.....(ii)},$$
$$c\ (s_3c_2 - s_4c_1) + c_6(s_4c_2 - s_3c_1) = 0 \text{.....(iii)},$$
$$c_5(c_1c_4 - s_2s_3) + c_6(s_1s_3 - c_2c_4) = 0 \text{.....(iv)},$$
$$s_5(s_3c_2 - s_4c_1) + c_6(c_1c_3 - c_2c_4) = 0 \text{.....(v)},$$
$$s_5(c_1c_4 - s_2s_3) + c_6(s_2c_4 - s_3c_1) = 0 \text{.....(vi)},$$
$$s_6(s_3c_1 - s_4c_2) + c_5(c_1c_4 - c_2c_3) = 0 \text{.....(vii)},$$
$$s_6(c_2c_4 - s_1s_3) + c_5(s_3c_2 - s_1c_4) = 0 \text{.....(viii)},$$
$$s_6(s_3c_1 - s_4c_2) + c_5(c_1c_3 - c_2c_4) = 0 \text{.....(ix)},$$
$$s_5(c_1c_2 - s_3s_4) + c_5(s_2c_1 - s_3c_4) = 0 \text{.....(x)},$$
$$s_5(s_1c_2 - s_3c_4) + c_5(c_1c_2 - c_3c_4) = 0 \text{.....(xi)},$$
$$s_5(c_2c_4 - s_1s_3) + c_5(s_2c_4 - s_3c_1) = 0 \text{.....(xii)},$$
$$s_6(s_3c_2 - s_4c_1) + c_6(c_1c_4 - c_2c_3) = 0 \text{.....(xiii)},$$
$$s_6(c_1c_2 - s_3s_4) + c_6(s_3c_4 - s_1c_2) = 0 \text{.....(xiv)},$$
$$s_6(s_2c_1 - s_3c_4) + c_6(c_3c_4 - c_1c_2) = 0 \text{.....(xv)},$$
$$s_6(c_1c_4 - s_2s_3) + c_6(s_3c_2 - s_1c_4) = 0 \text{.....(xvi)},$$

$$s_5(s_3c_4 - s_4c_3) + c_1c_3 - c_2c_4 = 0 \text{.....(xvii)},$$
$$s_5(c_2c_4 - s_3s_4) + s_2c_1 - s_3c_4 = 0 \text{.....(xviii)},$$
$$s_5(s_1c_3 - s_3c_1) + c_1c_2 - c_3c_4 = 0 \text{.....(xix)},$$
$$s_5(c_1c_2 - s_1s_2) + s_2c_4 - s_3c_1 = 0 \text{.....(xx)},$$
$$s_6(s_3c_4 - s_4c_3) + c_1c_4 - c_2c_3 = 0 \text{.....(xxi)},$$
$$s_6(c_1c_4 - s_1s_4) + s_3c_4 - s_1c_2 = 0 \text{.....(xxii)},$$
$$s_6(s_3c_2 - s_2c_3) + c_1c_2 - c_3c_4 = 0 \text{.....(xxiii)},$$
$$s_6(c_1c_3 - s_1s_3) + s_3c_2 - s_1c_4 = 0 \text{.....(xxiv)},$$
$$c_5(s_3c_4 - s_4c_3) + s_4c_2 - s_3c_1 = 0 \text{.....(xxv)},$$
$$c_5(s_1c_3 - s_3c_1) + s_3c_4 - s_1c_2 = 0 \text{.....(xxvi)},$$
$$c_5(c_2c_4 - s_2s_4) + s_3s_4 - c_1c_2 = 0 \text{.....(xxvii)},$$
$$c_5(c_1c_2 - s_1s_2) + s_1s_3 - c_2c_4 = 0 \text{.....(xxviii)},$$
$$c_6(s_3c_4 - s_4c_3) + s_4c_1 - s_3c_2 = 0 \text{.....(xxix)},$$
$$c_6(s_2c_3 - s_3c_2) + s_2c_1 - s_3c_4 = 0 \text{.....(xxx)},$$
$$c_6(c_1c_4 - s_1s_4) + s_3s_4 - c_1c_2 = 0 \text{.....(xxxi)},$$
$$c_6(c_1c_2 - s_1s_2) + s_2s_3 - c_1c_4 = 0 \text{.....(xxxii)}.$$

§4. The systems (1), ...(32) and (i), ...(xxxii) may be proved very easily by means of the following formulæ, in which

p denotes $\sin(A+B-C)$,

and q „ $\cos(A+B-C)$:

$$\left.\begin{aligned} s_5p &= s_1s_3 + s_2s_4 \\ &= c_2c_4 - c_1c_3 \end{aligned}\right\}, \qquad \left.\begin{aligned} s_5q &= s_1c_3 + s_4c_2 \\ &= s_3c_1 - s_2c_4 \end{aligned}\right\},$$

$$\left.\begin{aligned} -s_6p &= s_1s_4 + s_2s_3 \\ &= c_1c_4 - c_2c_3 \end{aligned}\right\}, \qquad \left.\begin{aligned} -s_6q &= s_2c_3 + s_4c_1 \\ &= s_3c_2 - s_1c_4 \end{aligned}\right\},$$

$$\left.\begin{aligned} c_5p &= s_1c_3 + s_2c_4 \\ &= s_3c_1 - s_4c_2 \end{aligned}\right\}, \qquad \left.\begin{aligned} c_5q &= s_2s_4 + c_1c_3 \\ &= c_2c_4 - s_1s_3 \end{aligned}\right\},$$

$$\left.\begin{aligned} c_6p &= s_1c_4 + s_2c_3 \\ &= s_3c_2 - s_4c_1 \end{aligned}\right\}, \qquad \left.\begin{aligned} c_6q &= s_1s_4 + c_2c_3 \\ &= c_1c_4 - s_2s_3 \end{aligned}\right\},$$

$$\left.\begin{aligned} p &= s_1c_2 + s_2c_1 \\ &= s_3c_4 - s_4c_3 \end{aligned}\right\}, \qquad \left.\begin{aligned} q &= s_3s_4 + c_3c_4 \\ &= c_1c_2 - s_1s_2 \end{aligned}\right\},$$

$$\left.\begin{aligned} s_5c_6 &= s_1c_2 + s_4c_3 \\ &= s_3c_4 - s_2c_1 \end{aligned}\right\}, \qquad \left.\begin{aligned} -s_6c_5 &= s_2c_1 + s_4c_3 \\ &= s_3c_4 - s_1c_2 \end{aligned}\right\},$$

$$\left.\begin{aligned} -s_5s_6 &= s_1s_2 + s_3s_4 \\ &= c_1c_2 - c_3c_4 \end{aligned}\right\}, \qquad \left.\begin{aligned} c_5c_6 &= s_1s_2 + c_3c_4 \\ &= c_1c_2 - s_3s_4 \end{aligned}\right\},$$

$$\left.\begin{aligned} s_5 &= s_1c_4 + s_4c_1 \\ &= s_3c_2 - s_2c_3 \end{aligned}\right\}, \qquad \left.\begin{aligned} c_5 &= s_2s_3 + c_2c_3 \\ &= c_1c_4 - s_1s_4 \end{aligned}\right\},$$

$$\left.\begin{aligned} -s_6 &= s_2c_4 + s_4c_2 \\ &= s_3c_1 - s_1c_3 \end{aligned}\right\}, \qquad \left.\begin{aligned} c_6 &= s_1s_3 + c_1c_3 \\ &= c_3c_4 - s_3s_4 \end{aligned}\right\}.$$

§ 5. Of the six arguments to which the suffixes refer, only three are independent, and if the first three A, $B-C$, B be denoted by α, β, γ, the last three will be denoted by $\gamma-\beta-\alpha$, $\gamma-\beta$, $\alpha-\gamma$, so that the scheme may be written

1	α
2	β
3	γ
4	$\gamma-\beta-\alpha$
5	$\gamma-\beta$
6	$\alpha-\gamma$

If the letters α, β be interchanged, the scheme becomes

1	β
2	α
3	γ
4	$\gamma-\alpha-\beta$
5	$\gamma-\alpha$
6	$\beta-\gamma$

which may also be derived from the original scheme by interchanging the arguments 1, 2, and by interchanging the arguments 5, 6 and changing their signs. It thus appears that, keeping the arguments 3, 4 fixed, we may interchange the arguments 1, 2 if we also interchange, with change of sign, the arguments 5, 6. We may therefore transform the formulæ as follows:

I. Keep 3, 4 fixed; interchange with change of sign 5, 6 and without change of sign 1, 2.

II. Keep 5, 6 fixed; interchange with change of sign 1, 2 and without change of sign 3, 4.

III. Keep 1, 2 fixed; interchange with change of sign 3, 4 and without change of sign 5, 6.

A change of sign in the argument only affects the sine, so that, for example, the interchange with change of sign of 5, 6 is equivalent to the interchange of c_5, c_6 and the substitution of $-s_5$ for s_6 and of $-s_6$ for s_5.

The application of the transformations (I), (II), (III) to (1), ...(32) leads to no new formulæ. This system when transformed by (I) reproduces itself. Of the transformed formulæ obtained by the application of either (II) or (III), ten belong to the system (1), ...(32), ten belong to the system (i), ...(xxxii), and twelve belong to neither system, one of the binomial expressions involved having the positive sign and the other the negative sign; but in the case of both transformations, the system of 32 formulæ reproduces itself exactly, if the binomials with negative sign be replaced by their equivalents with positive sign, given by the equations in § 2.

§ 6. The formulæ have been written so that the binomial expressions involve the suffixes 1, 2, 3, 4 and the factors involve the suffixes 5, 6; it is scarcely necessary to observe that all the suffixes may be changed cyclically, viz. we may replace 1, 2, 3, 4, 5, 6 by 3, 4, 5, 6, 1, 2 or 5, 6, 1, 2, 3, 4 respectively, so that the factors may involve the suffixes 1, 2 or 3, 4. Of course the formulæ transform into one another by the substitution of $\frac{1}{2}\pi - A$ for A, &c., and other changes of the arguments.

The system (1), ...(32) contains all the formulæ of the forms included in it that can be derived from the equations in § 4, except those obtained by combining among themselves the last four, such as, for example,

$$s_5(s_2c_4 + s_4c_2) + s_6(s_1c_4 + s_4c_1) = 0,$$

which is equivalent to $s_5s_6 - s_6s_5 = 0$. These were considered too elementary to be worth writing down. I may add that the formulæ (1), ...(32) were originally derived from a corresponding system of elliptic function formulæ.

The equation (1), which is the only one in which the same function (sine or cosine) is involved throughout, is the well known identity

$$\sin B \sin C \sin(B-C) + \sin C \sin A \sin(C-A) + \sin A \sin B \sin(A-B) + \sin(B-C)\sin(C-A)\sin(A-B) = 0,$$

and the next fifteen formulæ, with their transformations, may be regarded as completing the group to which it belongs.

INSIGNIORES ORBITÆ COMETARUM PROPRIETATES.

By *C. Taylor, M.A.*

(Continued from vol. IX. page 164.)

IT remains to give the proof* by which Lambert's theorem is established at once for the ellipse and the hyperbola, and by implication for the parabola regarded as a limiting form of ellipse. The figure is drawn to suit the ellipse, but the proof is equally applicable to the hyperbola.

IV.

1. *The area of any focal sector SQQ' of a central conic divided by the square root of its latus rectum may be expressed in terms of QQ', $SQ+SQ'$, and the transverse axis, and is independent of the magnitude of the conjugate axis* (fig. 4).

In an ellipse whose axes are AA' and BB' take any diameter PCP', and let PS meet the conjugate diameter CD in K and the tangent at P' in H, so that PK is equal to CA and PH to AA'.

With S as focus and P and H as vertices describe a second ellipse, and let KL be its semi-axis conjugate. Then if S' be the further focus of the first ellipse,

$$CD^2 = SP.PS' = SP.SH = KL^2.$$

(i) Let QOQ' be any double ordinate to PP', let it meet PH in o, and let qoq' be a double ordinate to PH in the second ellipse. Then by the nature of ordinates,

$$QO^2 : CD^2 = PO.OP' : CP^2 = Po.oH : KP^2$$
$$= qo^2 : KL^2,$$

or QO is equal to qo and QQ' to qq'.

(ii) And since $SK : CN = CS : CA\dagger = CS : PK$,

and $CN : CM = CP : CO = PK : Ko$;

therefore $SK : CM = CS : Ko$,

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* See *Sectio* IV. pp. 102–109. He treats the parabola as an ellipse of infinite major axis on p. 126.

† Because $SK = PK - SP = CA - SP = e.CN$.

or in terms of the eccentricities e and e',

$$e.CM = e'.Ko;$$

and therefore

$$SQ + SQ' = AA' + 2e.CM = PH + 2e'.Ko$$
$$= Sq + Sq'.$$

(iii) Let a line through S at right angles to QQ' meet QQ' in m and the tangent at P in n; then will the equal chords QQ' and qq' in any two consecutive positions cut off elements of area which, being as their own breadths, are in the *constant ratio* of Sm to So; and this is also the ratio of the triangles SQQ' and Sqq' on equal bases.

Therefore the whole segments $QP'Q'$ and qHq', as also the sectors SQQ' and Sqq', are as Sm to So, or as Sn to SP, or as CB to CD or KL; that is to say (the transverse axes being equal) the sectors SQQ' and Sqq' are in the subduplicate ratio of the latera recta of their ellipses.

(iv) Lastly (calling the second ellipse described as above the *auxiliary ellipse*) if any two ellipses having equal major axes be placed so as to have a common focal vector SP, they will have the same auxiliary ellipse; and it is inferred by comparing each of the two ellipses with their auxiliary ellipse that their corresponding focal sectors SQQ' will be in the subduplicate ratio of their latera recta provided that QQ' and $SQ + SQ'$ be equal in both. Hence, when the three lengths QQ', $SQ + SQ'$, AA' are given the area of the focal sector SQQ' divided by the square root of the latus rectum is determined.

2. The result may also be arrived at more directly, without the intervention of the auxiliary ellipse, as follows.

In any two ellipses (or hyperbolas) on equal transverse axes take equal focal vectors SP—first taking SP at random in the less eccentric ellipse and then finding an equal SP in the other—and let chords QQ' be drawn parallel to the tangents at P in both and so as to make equal intercepts on the equal focal vectors. Then it may be shewn, very much as in the preceding proof, that the chords QQ' are equal in both ellipses, that $SQ + SQ'$ is of the same length in both, and lastly that the focal sectors SQQ' are in the subduplicate ratio of the latera recta of their ellipses.

3. *Fourth method of determining the formula for the time of describing any arc $QP'Q'$ of the ellipse.*

It suffices to determine the time of describing the arc qq' of the auxiliary ellipse, which is the same as the time of describing the arc QQ'. If now α be eccentric angle of q, it is obvious that the sectorial area Sqq' is equal to

$$\sqrt{\left(\frac{CB^2}{CA}\right)}.CA^{\frac{1}{2}}(\alpha + e.\sin\alpha);$$

and that $$\frac{Sq+Sq'}{AA'} = \frac{2Sq}{AA'} = 1 + e.\cos\alpha,$$

and $$\frac{qq'}{AA'} = \frac{b}{a}\sin\alpha.$$

To reduce the result to the same form as in III. §5 (vol. IX. 162) write $\sin\beta$ for $\frac{b}{a}$ or $\sqrt{(1-e^2)}$, and $-\cos\beta$ for e. Then we have as before,

$$nt = 2\alpha - 2\sin\alpha.\cos\beta = \{\beta+\alpha-\sin(\beta+\alpha)\} - \{\beta-\alpha-(\sin\beta-\alpha)\};$$

together with the relations

$$\frac{Sq+Sq'}{AA'} = 1-\cos\alpha.\cos\beta;\quad \frac{qq'}{AA'} = \sin\alpha.\sin\beta;$$

or $$\frac{SQ+SQ'\pm QQ'}{AA'} = \frac{Sq+Sq'\pm qq'}{AA'} = 1-\cos(\beta\pm\alpha).$$

Thus the time of the describing the arc qq' (or QQ') is again determined as a function of QQ', $SQ+SQ'$, and AA'.

4. *The Hyperbola.*

The formula for the hyperbola is given in terms of hyperbolic sines and cosines in vol. VII. 99. It may also be obtained by integrating the equation for motion under the hyperbolic law in a flat hyperbola or *straight line*, as was done for the ellipse in III. §6 (vol. IX. 163).

5. Lambert's theorem was discussed at considerable length by Lexell in the first volume of the *Nova Acta Academiæ Scientiarum Imperialis Petropolitanæ*, pp. (140)–(183), 1787, in a paper dated, May 1784. He refers to Lagrange's proof by the integral calculus in the ninth volume of the *Nova Acta* of the Berlin Academy, and then adds:

"Hanc autem demonstrationem expendenti mihi, quum ista mox se obtulerit meditatio, eandem demonstrationem

paucis immutatis ad sectores quoque hyperbolicos circa focos hyperbolarum descriptos adplicari posse, istud argumentum ex instituto tractare in animum induxi."

He remarks further that he has in the first instance treated the theorem "methodo veteribus Geometris vsitata," since Lambert's proofs "plerumque per calculum procedant;" and that he afterwards proves it by an analysis, not only for the ellipse but also for the hyperbola. This claim of Lexell to have *extended* the theorem to the hyperbola is admitted by Chasles, who remarks: "La propriété de l'ellipse qui est le fondement de ce théorème convient aussi aux secteurs de l'hyperbole, ainsi que l'a démontré, par de simples considérations de Géométrie le célèbre Lexell" (*Aperçu historique*, p. 187, ed. 2).

Lexell first gives at great length the equivalent of the proof in IV. § 1. (i)–(iii), but does not see how to generalize the result in the manner indicated in § (iv).* Accordingly he remarks that the proof is restricted to the case in which the chord qq' of one of the segments is a principal ordinate, and then proceeds to prove the proposition over again in a more general way. His whole proof, including some lemmas which might have been taken for granted, thus fills seven closely packed quarto pages,† exclusive of diagrams. In the course of it he makes some use of trigonometrical notation. In the same article he gives "nonnullas egregias Sectionum Conicarum proprietates," the principal which is said to be the property, that a chord MN of an ellipse inclined at a given angle to the axis varies as $FM - FN$, where F is the focus. This of course follows at once from the property of the focus and directrix, with which Lexell should have been well acquainted, as it was to be found in Newton and in more than one treatise upon conics, in one of which it had been used as the defining property.

As regards his supposed extension of Lambert's theorem to the hyperbola, I have only to repeat that Lambert himself had so extended it. Compare the following passages from his work:

a. Writing on the parabola, he remarks by the way, "Ceterum hic notabimus simile quid obtineri posse si orbita fuerit sectio conica quæcunque, atque debitæ fiant mutationes & limitationes" (p. 29).

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* Lambert himself knew how to generalize his results. See §§ 95, 185 (pp. 45, 111).

† See pages (141)–(146) and (149), (150).

b. Having expressed t for the ellipse in the form of a series obtained by integration (p. 125), he remarks, "Similiter si FB sit hyperbolae axis tranversus, tempus lapsus hyperbolici cometae ex m in solem F erit, $t =$ &c. Unde facile construetur *scala celeritatum hyperbolicarum*, scalae ellipticarum & parabolicarum similis" (p. 126).

From these and similar passages it is evident that Lambert had not overlooked the case of the hyperbola.

ON THE MOTION OF A FRICTIONLESS LIQUID IN A ROTATING SECTOR.

By *A. G. Greenhill, M.A.*

If the sector be bounded by the cylinder $r = a$, and the planes $\theta = \pm\alpha$, then, ψ denoting the stream function and ϕ the velocity function (*Messenger of Mathematics*, vol. VIII. p. 89),

$$\psi + i\phi = \tfrac{1}{2}\omega \frac{a^2x^2}{\cos 2\alpha}$$

$$- \frac{\omega a^2}{2\alpha}\left(x^2\int_0^{\frac{\pi}{2\alpha}} \frac{x^{\frac{\pi}{2\alpha}-3}dx}{1+x^{\frac{\pi}{\alpha}}} - 2\int_0^{\frac{\pi}{2\alpha}} \frac{x^{\frac{\pi}{2\alpha}-1}dx}{1+x^{\frac{\pi}{\alpha}}} + x^{-2}\int_0^{\frac{\pi}{2\alpha}} \frac{x^{\frac{\pi}{2\alpha}+1}dx}{1+x^{\frac{\pi}{\alpha}}}\right),$$

where $$x = \frac{r}{a}e^{i\theta}.$$

Put $\frac{\pi}{2\alpha} = \frac{m}{n}$, where m and n are integers, and $x = z^n$; then

$$\psi + i\phi = \tfrac{1}{2}\omega \frac{a^2x^2}{\cos 2\alpha}$$

$$- \frac{m\omega a^2}{\pi}\left(x^2\int_0 \frac{z^{m-2n-1}}{1+z^{2m}}dz - 2\int_0 \frac{z^{m-1}}{1+z^{2m}}dz + x^{-2}\int_0 \frac{z^{m+2n-1}}{1+z^{2m}}dz\right).$$

I. Suppose the angle of the sector less than a right angle, then $m - 2n$ is positive, and

$$\frac{z^{m-2n-1}}{1+z^{2m}} = -\frac{1}{2m}\sum_{r=1}^{r=2m} \frac{e^{(m-2n-1)(2r+1)\frac{i\pi}{2m}}}{1 - ze^{-(2r+1)\frac{i\pi}{2m}}},$$

and therefore

$$\int_0 \frac{z^{m-2n-1}}{1+z^{2m}}dz = \frac{1}{2m}\Sigma e^{(m-2n)(2r+1)\frac{i\pi}{2m}} \log\{1 - ze^{-(2r+1)\frac{i\pi}{2m}}\},$$

and similarly for the other integrals.

Therefore

$$\psi + i\dot{\phi} = \tfrac{1}{2}\omega \frac{a^2x^2}{\cos 2\alpha}$$

$$- \frac{\omega a^2}{2\pi} \sum_{r=1}^{r=2m} \left\{x^2 e^{(m-2n)(2r+1)\frac{i\pi}{2m}} - 2e^{(2r+1)\frac{i\pi}{2}} + x^2 e^{(m+2n)(2r+1)\frac{i\pi}{2m}}\right\}$$

$$\times \log\left\{1 - ze^{-(2r+1)\frac{i\pi}{2m}}\right\}$$

$$= \tfrac{1}{2}\omega\left(r^2\cos 2\theta + ir^2\sin 2\theta\right)$$

$$- \frac{\omega a^2}{2\pi} \Sigma e^{(2r+1)\frac{i\pi}{2}} \left\{\frac{r^2}{a^2} e^{2i\theta-(2r+1)\frac{in\pi}{m}} - 2 + \frac{a^2}{r^2} e^{-2i\theta+(2r+1)\frac{in\pi}{m}}\right\}$$

$$\times \log\left\{1 - \left(\frac{r}{a}\right)^{\frac{1}{n}} e^{i\frac{\theta}{n}-(2r+1)\frac{i\pi}{2m}}\right\}$$

$$= \tfrac{1}{2}\omega \frac{r^2\cos 2\theta + ir^2\sin 2\theta}{\cos 2\alpha}$$

$$- i\frac{\omega a^2}{2\pi} \Sigma(-1)^r \left[\left(\frac{r^2}{a^2} + \frac{a^2}{r^2}\right)\cos\left\{2\theta - (2r+1)\frac{n\pi}{m}\right\} - 2\right.$$

$$\left. + i\left(\frac{r^2}{a^2} - \frac{a^2}{r^2}\right)\sin\left\{2\theta - (2r+1)\frac{n\pi}{m}\right\}\right]$$

$$\times \left[\tfrac{1}{2}\log\left\{1 - 2\left(\frac{r}{a}\right)^{\frac{1}{n}}\cos\left(\frac{\theta}{n} - (2r+1)\frac{\pi}{2m}\right) + \left(\frac{r}{a}\right)^{\frac{2}{n}}\right\}\right.$$

$$\left. - i\tan^{-1}\frac{\left(\frac{r}{a}\right)^{\frac{1}{n}}\sin\left\{\frac{\theta}{n} - (2r+1)\frac{\pi}{2m}\right\}}{1 - \left(\frac{r}{a}\right)^{\frac{1}{n}}\cos\left\{\frac{\theta}{n} - (2r+1)\frac{\pi}{2m}\right\}}\right],$$

and therefore

$$\psi = \tfrac{1}{2}\omega r^2 \frac{\cos 2\theta}{\cos 2\alpha}$$

$$+ \frac{\omega a^2}{2\pi} \Sigma(-1)^r \left(\frac{r^2}{a^2} - \frac{a^2}{r^2}\right)\sin\left\{2\theta - (2r+1)\frac{n\pi}{m}\right\}$$

$$\times \log\left\{1 - 2\left(\frac{r}{a}\right)^{\frac{1}{n}}\cos\left\{\frac{\theta}{n} - (2r+1)\frac{\pi}{2m}\right\} + \left(\frac{r}{a}\right)^{\frac{2}{n}}\right\}$$

$$- \frac{\omega a^2}{2\pi} \Sigma(-1)^r \left[\left(\frac{r^2}{a^2} + \frac{a^2}{r^2}\right)\cos\left\{2\theta - (2r+1)\frac{n\pi}{m}\right\} - 2\right]$$

$$\times \tan^{-1}\frac{\left(\frac{r}{a}\right)^{\frac{1}{n}}\sin\left\{\frac{\theta}{n} - (2r+1)\frac{\pi}{2m}\right\}}{1 - \left(\frac{r}{a}\right)^{\frac{1}{n}}\cos\left\{\frac{\theta}{n} - (2r+1)\frac{\pi}{2m}\right\}},$$

$$\phi = \tfrac{1}{2}\omega r^2 \sin 2\theta$$

$$- \frac{\omega a^2}{4\pi} \Sigma (-1)^r \left[\left(\frac{r^2}{a^2} + \frac{a^2}{r^2} \right) \cos \left\{ 2\theta - (2r+1)\frac{n\pi}{m} \right\} - 2 \right]$$

$$\times \log \left[1 - 2 \left(\frac{r}{a} \right)^{\frac{1}{n}} \cos \left\{ \frac{\theta}{n} - (2r+1)\frac{\pi}{2m} \right\} + \left(\frac{r}{a} \right)^{\frac{2}{n}} \right]$$

$$- \frac{\omega a^2}{2\pi} \Sigma (-1)^r \left(\frac{r^2}{a^2} - \frac{a^2}{r^2} \right) \sin \left\{ 2\theta - (2r+1)\frac{n\pi}{m} \right\}$$

$$\times \tan^{-1} \frac{\left(\frac{r}{a} \right)^{\frac{1}{n}} \sin \left\{ \frac{\theta}{n} - (2r+1)\frac{\pi}{2m} \right\}}{1 - \left(\frac{r}{a} \right)^{\frac{1}{n}} \cos \left\{ \frac{\theta}{n} - (2r+1)\frac{\pi}{2m} \right\}}.$$

II. Suppose the angle of the sector greater than a right angle, but less than three right angles; then $m - 2n$ is negative, but $3m - 2n$ is positive, and

$$\int \frac{z^{m-2n-1}}{1+z^{2m}} = \int \left(z^{m-2n-1} - \frac{z^{3m-2n-1}}{1+z^{2m}} \right) dz$$

$$= \frac{z^{m-2n}}{m-2n} - \int \frac{z^{3m-2n-1}}{1+z^{2m}} dz$$

$$\int \frac{z^{-m+2n-1}}{1+z^{2m}} dz = \int \left(z^{-m+2n-1} - \frac{z^{-m+2n-1}}{1+z^{2m}} \right) dz$$

$$= - \frac{z^{-m+2n}}{m-2n} - \int \frac{z^{-m+2n-1}}{1+z^{2m}} dz,$$

and

$$\psi + i\phi = \tfrac{1}{2}\omega \frac{a^2 x^2}{\cos 2\alpha}$$

$$- \frac{\omega a^2}{\pi} \frac{z^m - z^{-m}}{1 - 2\frac{n}{m}}$$

$$+ \frac{m\omega a^2}{\pi} \left(x^2 \int \frac{z^{3m-2n-1}}{1+z^{2m}} dz + 2 \int \frac{z^{m-1}}{1+z^{2m}} dz + x^{-2} \int \frac{z^{-m+2n-1}}{1+z^{2m}} dz \right)$$

$$= \tfrac{1}{2}\omega\,(r^2\cos 2\theta + ir^2\sin 2\theta)$$

$$- \frac{\omega a^2}{\pi}\,\frac{\left\{\left(\frac{r}{a}\right)^{\frac{m}{n}} - \left(\frac{a}{r}\right)^{\frac{m}{n}}\right\}\cos\frac{m}{n}\theta + i\left\{\left(\frac{r}{a}\right)^{\frac{m}{n}} + \left(\frac{a}{r}\right)^{\frac{m}{n}}\right\}\sin\frac{m}{n}\theta}{1-2\,\frac{n}{m}}$$

$$+ \frac{\omega a^2}{2\pi}\,\Sigma\left\{x^2 e^{(3m-2n)(2r+1)\frac{i\pi}{2m}} + 2e^{(2r+1)\frac{i\pi}{2}} + \frac{1}{x^2}\,e^{(-m+2n)(2r+1)\frac{i\pi}{2m}}\right\}$$

$$\times \log\left\{1 - ze^{-(2r+1)\frac{i\pi}{2m}}\right\}$$

$$= \tfrac{1}{2}\omega\,(r^2\cos 2\theta + ir^2\sin 2\theta)$$

$$- \frac{\omega a^2}{\pi}\,\frac{\left\{\left(\frac{r}{a}\right)^{\frac{m}{n}} - \left(\frac{a}{r}\right)^{\frac{m}{n}}\right\}\cos\frac{m}{n}\theta + i\left\{\left(\frac{r}{a}\right)^{\frac{m}{n}} + \left(\frac{a}{r}\right)^{\frac{m}{n}}\right\}\sin\frac{m}{n}\theta}{1-2\,\frac{n}{m}}$$

$$- \frac{\omega a^2}{2\pi}\,\Sigma\left\{x^2 e^{(m-2n)(2r+1)\frac{i\pi}{2m}} - 2e^{(2r+1)\frac{i\pi}{2}} + \frac{1}{x^2}\,e^{(m+2n)(2r+1)\frac{i\pi}{2m}}\right\}$$

$$\times \log\left\{1 - ze^{-(2r+1)\frac{i\pi}{2}}\right\};$$

and therefore the values of ψ and ϕ now only differ from the former values when the angle of the sector was less than a right angle by the addition of the terms

$$- \frac{\omega a^2}{\pi}\,\frac{\left(\frac{r}{a}\right)^{\frac{m}{n}} - \left(\frac{a}{r}\right)^{\frac{m}{n}}}{1-2\,\frac{n}{m}}\cos\frac{m}{n}\theta,$$

$$- \frac{\omega a^2}{\pi}\,\frac{\left(\frac{r}{a}\right)^{\frac{m}{n}} + \left(\frac{a}{r}\right)^{\frac{m}{n}}}{1-2\,\frac{n}{m}}\sin\frac{m}{n}\theta$$

respectively.

III. Suppose the angle of the sector greater than three right angles, then $3m-2n$ is negative, and

$$\int\frac{z^{m-2n-1}}{1+z^{2m}} = \int\left(z^{m-2n-1} - z^{3m-2n-1} + \frac{z^{5m-2n-1}}{1+z^{2m}}\right)dz$$

$$= \frac{z^{m-2n}}{m-2n} - \frac{z^{3m-2n}}{3m-2n} + \int_0\frac{z^{5m-2n-1}}{1+z^{2m}}\,dz,$$

$$\int \frac{z^{\frac{m}{n}+2m-1}}{1+z^{\frac{2m}{n}}}\,dz = \int\left(z^{-m+2m-1} - z^{-3m+n-1} + \frac{z^{-3m+2m-1}}{1+z^{\frac{2m}{n}}}\right)dz$$

$$= -\frac{z^{-m+2m}}{m-2n} + \frac{z^{-3m+2n}}{3m-2n} + \int \frac{z^{-3m+2m-1}}{1+z^{\frac{2m}{n}}}\,dz;$$

and $\psi + i\phi = \frac{1}{2}\omega \dfrac{a^2x^2}{\cos 2\alpha}$

$$-\frac{\omega a^2}{\pi}\,\frac{z^m - z^{-m}}{1-2\frac{n}{m}} + \frac{\omega a^2}{\pi}\,\frac{z^{3m} - z^{-3m}}{3-2\frac{n}{m}}$$

$$-\frac{m\omega a^2}{\pi}\left(x^2\int\frac{z^{5m-2m-1}}{1+z^{\frac{2m}{n}}}\,dz - 2\int\frac{z^{m-1}}{1+z^{\frac{2m}{n}}}\,dz + x^{-2}\int\frac{z^{-3m+2m-1}}{1+z^{\frac{2m}{n}}}\,dz\right)$$

$$= \tfrac{1}{2}\omega\,(r^2\cos 2\theta + ir^2\sin 2\theta)$$

$$-\frac{\omega a^2}{\pi}\,\frac{\left\{\left(\frac{r}{a}\right)^{\frac{m}{n}} - \left(\frac{a}{r}\right)^{\frac{m}{n}}\right\}\cos\frac{m}{n}\theta + i\left\{\left(\frac{r}{a}\right)^{\frac{m}{n}} + \left(\frac{a}{r}\right)^{\frac{m}{n}}\right\}\sin\frac{m}{n}\theta}{1-2\frac{n}{m}}$$

$$+\frac{\omega a^2}{\pi}\,\frac{\left\{\left(\frac{r}{a}\right)^{3\frac{m}{n}} - \left(\frac{a}{r}\right)^{3\frac{m}{n}}\right\}\cos 3\frac{m}{n}\theta + i\left\{\left(\frac{r}{a}\right)^{3\frac{m}{n}} + \left(\frac{a}{r}\right)^{3\frac{m}{n}}\right\}\sin 3\frac{m}{n}\theta}{3-2\frac{n}{m}}$$

$$-\frac{\omega a^2}{2\pi}\,\Sigma\left\{x^2 e^{(5m-2n)(2r+1)\frac{i\pi}{2m}} - 2e^{(2r+1)\frac{i\pi}{2}} + \frac{1}{x^2}e^{(-3m+2n)(2r+1)\frac{i\pi}{2m}}\right\}$$

$$\times \log\left\{1 - ze^{-(2r+1)\frac{i\pi}{2m}}\right\};$$

and the values of ψ and ϕ differ only from the values obtained when the angle of the sector was less than a right angle by the addition of the terms

$$-\frac{\omega a^2}{\pi}\,\frac{\left(\frac{r}{a}\right)^{\frac{m}{n}} - \left(\frac{a}{r}\right)^{\frac{m}{n}}}{1-2\frac{m}{n}}\cos\frac{m}{n}\theta + \frac{\omega a^2}{\pi}\,\frac{\left(\frac{r}{a}\right)^{3\frac{m}{n}} - \left(\frac{a}{r}\right)^{3\frac{m}{n}}}{3-2\frac{m}{n}}\cos 3\frac{m}{n}\theta,$$

$$-\frac{\omega a^2}{\pi}\,\frac{\left(\frac{r}{a}\right)^{\frac{m}{n}} + \left(\frac{a}{r}\right)^{\frac{m}{n}}}{1-2\frac{m}{n}}\sin\frac{m}{n}\theta + \frac{\omega a^2}{\pi}\,\frac{\left(\frac{r}{a}\right)^{3\frac{m}{n}} + \left(\frac{a}{r}\right)^{3\frac{m}{n}}}{3-2\frac{m}{n}}\sin 3\frac{m}{n}\theta$$

respectively.

If T denote the kinetic energy per unit length of the cylinder due to the angular velocity ω, ρ being the density of the liquid,

$$T = \tfrac{1}{2}\rho \int \phi \frac{d\phi}{dn}\, ds,$$

and when $r = a$, $\dfrac{d\phi}{dn} = 0$,

when $\theta = \alpha$, $\dfrac{d\phi}{dn} = \omega r$,

$\theta = -\alpha$, $\dfrac{d\phi}{dn} = -\omega r$,

and $\phi_\alpha = -\phi_{-\alpha}$;

therefore $T = \rho\omega \int_0^a \phi_\alpha r\, dr$;

and since $\phi_\alpha = \tfrac{1}{2}\omega r^2 \tan 2\alpha$

$$-\frac{\omega a^2}{2\alpha} \sum_{n=0}^{n=\infty} \left(\frac{r}{a}\right)^{(2n+1)\frac{\pi}{2\alpha}} \left\{ \frac{1}{(2n+1)\frac{\pi}{2\alpha} - 2} - \frac{2}{(2n+1)\frac{\pi}{2\alpha}} + \frac{1}{(2n+1)\frac{\pi}{2\alpha} + 2} \right\},$$

therefore $T = \tfrac{1}{8}\rho\omega^2 a^4 \tan 2\alpha$

$$-\frac{\rho\omega^2 a^4}{2\alpha} \Sigma \frac{1}{(2n+1)\frac{\pi}{2\alpha} + 2} \times \left\{ \frac{1}{(2n+1)\frac{\pi}{2\alpha} - 2} - \frac{2}{(2n+1)\frac{\pi}{2\alpha}} + \frac{1}{(2n+1)\frac{\pi}{2\alpha} + 2} \right\}$$

$$= \tfrac{1}{8}\rho\omega^2 a^4 \tan 2\alpha - \frac{\rho\omega^2 a^4}{8\alpha} \Sigma \left\{ \frac{1}{(2n+1)\frac{\pi}{2\alpha} - 2} - \frac{4}{(2n+1)\frac{\pi}{2\alpha}} + \frac{3}{(2n+1)\frac{\pi}{2\alpha} + 2} + \frac{4}{\left((2n+1)\frac{\pi}{2\alpha} + 2\right)^2} \right\}$$

$$= \tfrac{1}{8}\rho\omega^2 a^4 \tan 2\alpha - \frac{\rho\omega^2 a^4}{8\pi} \Sigma \left\{ \frac{1}{n+\frac{1}{2}-\left(\frac{\pi}{2\alpha}\right)^2} - \frac{4}{n+\frac{1}{2}} + \frac{3}{n+\frac{1}{2}+\frac{2\alpha}{\pi}} + \frac{\frac{4\alpha}{\pi}}{\left(n+\frac{1}{2}+\frac{2\alpha}{\pi}\right)^2} \right\}$$

$$= \tfrac{1}{8}\rho\omega^2 a^4 \tan 2\alpha + \frac{\rho\omega^2 a^4}{8\pi} \left\{ \psi\left(\tfrac{1}{2}-\frac{2\alpha}{\pi}\right) - 4\psi\left(\tfrac{1}{2}\right) + 3\psi\left(\tfrac{1}{2}+\frac{2\alpha}{\pi}\right) - \frac{4\alpha}{\pi}\psi'\left(\tfrac{1}{2}+\frac{2\alpha}{\pi}\right) \right\},$$

where $\psi(x) = \dfrac{d}{dx}\log\Gamma(x)$ (Bertrand, *Calcul. Intégral*, p. 252), and since

$$\psi\left(\tfrac{1}{2}+x\right) - \psi\left(\tfrac{1}{2}-x\right) = \pi \tan \pi x,$$

therefore

$$T = \frac{\rho\omega^2 a^4}{2\pi} \left\{ \psi\left(\tfrac{1}{2}+\frac{2\alpha}{\pi}\right) - \psi\left(\tfrac{1}{2}\right) - \frac{\alpha}{\pi}\psi'\left(\tfrac{1}{2}+\frac{2\alpha}{\pi}\right) \right\}.$$

If k denote the effective radius of gyration of the liquid filling the sector,

$$T = \tfrac{1}{2}\rho\omega^2 a^2 \alpha k^2,$$

and therefore

$$\frac{k^2}{a^2} = \frac{1}{\pi\alpha} \left\{ \psi\left(\tfrac{1}{2}+\frac{2\alpha}{\pi}\right) - \psi\left(\tfrac{1}{2}\right) \right\} - \frac{1}{\pi^2}\psi'\left(\tfrac{1}{2}+\frac{2\alpha}{\pi}\right).$$

For instance, in the semicircle, $\alpha = \frac{1}{2}\pi$, and

$$\frac{k^2}{a^2} = \frac{2}{\pi^2}\left\{\psi\left(\tfrac{3}{2}\right) - \psi\left(\tfrac{1}{2}\right)\right\} - \frac{1}{\pi^2}\psi'\left(\tfrac{3}{2}\right)$$

$$= \frac{4}{\pi^2} - \frac{4}{\pi^2}\left(\frac{\pi^2}{8} - 1\right)$$

$$= \frac{8}{\pi^2} - \tfrac{1}{2} = .31056.$$

In the quadrant, $\alpha = \frac{1}{4}\pi$,

$$\frac{k^2}{a^2} = \frac{4}{\pi^2}\left\{\psi(1) - \psi\left(\tfrac{1}{2}\right)\right\} - \frac{1}{\pi^2}\psi'(1)$$

$$= \frac{8}{\pi^2}\log 2 - \tfrac{1}{6} = .39518.$$

ON THE CRITERIA FOR DETERMINING THE NATURE OF THE CONIC REPRESENTED BY THE GENERAL EQUATION IN AREAL COORDINATES.

By *Harry Hart, M.A.*

LET the equation be

$$ux^2 + vy^2 + wz^2 + 2u'yz + 2v'zx + 2w'xy = 0.$$

Then, putting
$$v + w - 2u' = 2\alpha,$$
$$w + u - 2v' = 2\beta,$$
$$u + v - 2w' = 2\gamma,$$

the equation to determine the lengths of the semi-axes of the conic (Ferrers' Trilinear Coordinates, edit. p. 95) may be written

$$\begin{vmatrix} \frac{s}{r^2}\frac{\cos A}{a}, & \gamma, & \beta, & 1 \\ \gamma, & \frac{s}{r^2}\frac{\cos B}{b}, & \alpha, & 1 \\ \beta, & \alpha, & \frac{s}{r^2}\frac{\cos C}{c}, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix} = 0,$$

which, expanded, becomes

$$\frac{1}{r^4}\left\{\frac{2\Delta s}{abc}\right\}^2 - \frac{2}{r^2}\left\{\frac{2\Delta s}{abc}\right\}(\alpha\cot A + \beta\cot B + \gamma\cot C)$$
$$-\{\alpha^2 + \beta^2 + \gamma^2 - 2\beta\gamma - 2\gamma\alpha - 2\alpha\beta\} = 0,$$

where $s = \frac{abcD}{B}$, $D =$ the discriminant $\begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ v', & u', & w \end{vmatrix}$,

and $$B = \begin{vmatrix} u, & w', & v', & 1 \\ w', & v, & u', & 1 \\ v', & u', & w, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix} = \begin{vmatrix} 0, & \gamma, & \beta, & 1 \\ \gamma, & 0, & \alpha, & 1 \\ \beta, & \alpha, & 0, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix}$$
$$= \alpha^2 + \beta^2 + \gamma^2 - 2\beta\gamma - 2\gamma\alpha - 2\alpha\beta,$$

substituting for s its value, we have

$$\left\{\frac{r^2}{\Delta D}\right\}^2 + 4\left\{\frac{r^2}{\Delta D}\right\}\frac{\alpha\cot A+\beta\cot B+\gamma\cot C}{B^2} - \frac{4}{B^2} = 0,$$

or $$r^4 + 4r^2\Delta D\frac{\alpha\cot A+\beta\cot B+\gamma\cot C}{B^2} - \frac{4\Delta^2D^2}{B^2} = 0.$$

1. Suppose two of the quantities—α, β, say—of opposite sign to γ.

Then
$$\begin{aligned} B &= \alpha^2+\beta^2+\gamma^2-2\beta\gamma-2\gamma\alpha-2\alpha\beta \\ &= (\alpha-\beta+\gamma)^2-4\alpha\gamma \\ &= \text{a positive quantity,} \end{aligned}$$

thus, when α, β, γ are not all of the same sign, the conic is real and a hyperbola.

2. Suppose α, β, γ affected with the same sign, which we may consider positive.

Put $$\alpha=\xi^2,\ \beta=\eta^2,\ \gamma=\zeta^2;$$

then
$$\begin{aligned} B &= \xi^4+\eta^4+\zeta^4-2\eta^2\zeta^2-2\zeta^2\xi^2-2\xi^2\eta^2 \\ &= -(\xi+\eta+\zeta)(-\xi+\eta+\zeta)(\xi-\eta+\zeta)(\xi+\eta-\zeta), \end{aligned}$$

and the quadratic to determine the two values of r^2 becomes

$$r^4 + 4r^2\Delta\frac{D(\xi^2\cot A+\eta^2\cot B+\zeta^2\cot C)}{B^2}$$

$$+4\Delta^2D^2\{(\xi+\eta+\zeta)(-\xi+\eta+\zeta)(\xi-\eta+\zeta)(\xi+\eta-\zeta)\}^{-2}=0.$$

The conic is a parabola if $B=0$, that is, if any one of the three quantities = sum of the other two, a hyperbola if any one be less than the sum of the other two, and a rectangular hyperbola if further

$$\xi^2\cot A+\eta^2\cot B+\zeta^2\cot C=0,$$

and an ellipse if the sum of any two be greater than the third, and imaginary if in addition to this condition the discriminant D is of the same sign as

$$\xi^2\cot A+\eta^2\cot B+\zeta^2\cot C.$$

These conditions admit of a simple geometrical representation. Let ξ, η, ζ be the areal coordinates of a point P, and D, E, F the middle points of the sides of the triangle of reference. Then, confining our attention to positive values

of ξ, η, ζ, the conic will be an ellipse, hyperbola, or parabola, according as P lies within, without, or on the perimeter of the triangle DEF. It will be a circle if P is at the centre of the inscribed circle of the triangle of reference, and a rectangular hyperbola if P lies on the circle to which this triangle is self-conjugate.

Since $\xi^2 \cot A + \eta^2 \cot B + \zeta^2 \cot C$ is positive for points lying outside this last-named circle, it follows that when the general equation represents an ellipse

$\alpha \cot A + \beta \cot \beta + \gamma \cot C$ is of the same sign as α or β or γ,

and that therefore when B is negative, the conic will be imaginary or real according as the discriminant is, or is not, of the same sign as either of the expressions

$$v + w - 2u', \quad w + u - 2v', \quad u + v - 2w'.$$

ON SOME ELLIPTIC FUNCTION AND TRIGONOMETRICAL THEOREMS.

By *J. W. L. Glaisher.*

§1. Writing $S_1, S_2, \ldots S_6$ in place of $\operatorname{sn} A$, $\operatorname{sn}(B-C)$, $\ldots \operatorname{sn}(A-B)$ so that the suffixes and arguments are connected according to the scheme

1	A
2	$B-C$
3	B
4	$C-A$
5	C
6	$A-B$

the theorem given by Mr. M. M. U. Wilkinson, on p. 65, is

$$(S_1S_2 + S_3S_4 + S_5S_6)(S_1S_4S_6 + S_3S_6S_2 + S_5S_2S_4 + S_1S_3S_5)$$
$$= -S_2S_3S_5(S_2^2 + S_3^2 + S_5^2) - S_4S_5S_1(S_4^2 + S_5^2 + S_1^2)$$
$$- S_6S_1S_3(S_6^2 + S_1^2 + S_3^2) - S_2S_4S_6(S_2^2 + S_4^2 + S_6^2) \ldots (1).$$

Mr. Wilkinson's proof is as follows:

By ordinary multiplication the left-hand member of the equation

$$= S_1^2(S_2S_3S_5 + S_2S_4S_6) + S_3^2(S_4S_5S_1 + S_2S_4S_6) + S_5^2(S_6S_1S_3 + S_2S_4S_6)$$
$$+ S_2^2(S_4S_5S_1 + S_6S_1S_3) + S_4^2(S_6S_1S_3 + S_2S_3S_5) + S_6^2(S_2S_3S_5 + S_4S_5S_1);$$

and this on transforming each of the expressions in brackets by means of the formula

$$S_2S_3S_5 + S_4S_5S_1 + S_6S_1S_3 + S_2S_4S_6 = 0 \ldots\ldots\ldots (2),$$

becomes

$$-S_1^2(S_4S_5 + S_6S_3) - S_3^2(S_6S_1 + S_2S_5) - S_5^2(S_2S_3 + S_4S_1)$$
$$-S_2^2(S_3S_5 + S_4S_6) - S_4^2(S_5S_1 + S_6S_2) - S_6^2(S_1S_3 + S_2S_4),$$

which is in fact the right-hand member of (1).

If in (1), $S_1S_2 + S_3S_4 + S_5S_6$ be replaced by its equivalent, given by the formula

$$S_1S_2 + S_3S_4 + S_5S_6 + k^2S_1S_2S_3S_4S_5S_6 = 0 \ldots\ldots (3),$$

we have

$$S_2S_3S_5(S_2^2 + S_3^2 + S_5^2) + S_4S_5S_1(S_4^2 + S_5^2 + S_1^2)$$
$$+ S_6S_1S_3(S_6^2 + S_1^2 + S_3^2) + S_2S_4S_6(S_2^2 + S_4^2 + S_6^2)$$
$$= k^2S_1S_2S_3S_4S_5S_6(S_1S_4S_6 + S_3S_6S_2 + S_5S_2S_4 + S_1S_3S_5) \ldots (4).$$

§ 2. The proof in the last section may be presented in the following slightly different form.

In virtue of (2), we have

$$(S_1^2 + S_2^2 + S_3^2 + S_4^2 + S_5^2 + S_6^2)(S_2S_3S_5 + S_4S_5S_1 + S_6S_1S_3 + S_2S_4S_6) = 0,$$

and therefore

$$S_2S_3S_5(S_2^2 + S_3^2 + S_5^2) + S_4S_5S_1(S_4^2 + S_5^2 + S_1^2)$$
$$+ S_6S_1S_3(S_6^2 + S_1^2 + S_3^2) + S_2S_4S_6(S_2^2 + S_4^2 + S_6^2)$$
$$= -S_2S_3S_5(S_1^2 + S_4^2 + S_6^2) - S_4S_5S_1(S_3^2 + S_6^2 + S_2^2)$$
$$-S_6S_1S_3(S_5^2 + S_2^2 + S_4^2) - S_2S_4S_6(S_1^2 + S_3^2 + S_5^2)$$
$$= -(S_1S_2 + S_3S_4 + S_5S_6)(S_1S_4S_6 + S_3S_6S_2 + S_5S_2S_4 + S_1S_3S_5).$$

§ 3. The formula (1) thus depends upon the algebraical identity:

If a, b, c, f, g, h be any six quantities,

$$bcf(b^2 + c^2 + f^2) + cag(c^2 + a^2 + g^2)$$
$$+ abh(a^2 + b^2 + h^2) + fgh(f^2 + g^2 + h^2)$$
$$= (a^2 + b^2 + c^2 + f^2 + g^2 + h^2)(bcf + cag + abh + fgh)$$
$$-(af + bg + ch)(gha + hfb + fgc + abc) \ldots\ldots\ldots\ldots (5).$$

By interchanging a and f, b and g, c and h we deduce from (5) the identity

$$\begin{aligned}&ghα(g^2+h^2+a^2)+hfb(h^2+f^2+b^2)\\&+fgc(f^2+g^2+c^2)+abc(a^2+b^2+c^2)\\&=(a^2+b^2+c^2+f^2+g^2+h^2)(ghα+hfb+fgc+abc)\\&\qquad-(af+bg+ch)(bcf+cag+abh+fgh)\ldots\ldots(6).\end{aligned}$$

If we put

$$a, b, c, f, g, h = S_1, S_3, S_5, S_2, S_4, S_6,$$

so that $$bcf+cag+abh+fgh=0,$$

(5) gives (1), and from (6) we have

$$\begin{aligned}&S_1S_4S_6(S_1^2+S_4^2+S_6^2)+S_3S_6S_2(S_3^2+S_6^2+S_2^2)\\&+S_5S_2S_4(S_5^2+S_2^2+S_4^2)+S_1S_3S_5(S_1^2+S_3^2+S_5^2)\\&=(S_1^2+S_2^2+S_3^2+S_4^2+S_5^2+S_6^2)(S_1S_4S_6+S_3S_6S_2+S_5S_2S_4+S_1S_3S_5)\\&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ldots\ldots\ldots\ldots\ldots\ldots(7),\end{aligned}$$

whence, multiplying together the two factors which form the right-hand member,

$$\begin{aligned}&S_1S_4S_6(S_2^2+S_3^2+S_5^2)+S_3S_6S_2(S_4^2+S_5^2+S_1^2)\\&+S_5S_2S_4(S_6^2+S_1^2+S_3^2)+S_1S_3S_5(S_2^2+S_4^2+S_6^2)=0\ldots.(8).\end{aligned}$$

From (4) and (7) we obtain by division

$$\frac{\Sigma\{S_2S_3S_5(S_2^2+S_3^2+S_5^2)\}}{\Sigma\{S_1S_4S_6(S_1^2+S_4^2+S_6^2)\}}=k^2\frac{S_1S_2S_3S_4S_5S_6}{S_1^2+S_2^2+S_3^2+S_4^2+S_5^2+S_6^2}\ldots(9).$$

Corresponding to (8), we have

$$\begin{aligned}&S_2S_3S_5(S_1^2+S_4^2+S_6^2)+S_4S_5S_1(S_3^2+S_6^2+S_2^2)\\&+S_6S_1S_3(S_5^2+S_2^2+S_4^2)+S_2S_4S_6(S_1^2+S_3^2+S_5^2)\\&+k^2S_1S_2S_3S_4S_5S_6(S_1S_4S_6+S_3S_6S_2+S_5S_2S_4+S_1S_3S_5)=0\ldots(10),\end{aligned}$$

which is involved in the proof of (4).

§4. By putting k' for k, and iA, iB, iC in place of A, B, C we deduce from (2), (3), (4), (8), (9), by means of the formula

$$\operatorname{sn}(ui, k')=i\frac{\operatorname{sn}u}{\operatorname{cn}u},$$

the following equations, in which T_1, T_2, &c. denote $\frac{\mathrm{sn}\,A}{\mathrm{cn}\,A}$, $\frac{\mathrm{sn}\,(B-C)}{\mathrm{cn}\,(B-C)}$, &c.

$$T_2T_3T_5 + T_4T_5T_1 + T_6T_1T_3 + T_3T_4T_6 = 0 \ldots (11),$$

$$T_1T_2 + T_3T_4 + T_5T_6 + k^2T_1T_2T_3T_4T_5T_6 = 0 \ldots (12),$$

$$\begin{aligned} &T_2T_3T_5(T_2^2 + T_3^2 + T_5^2) + T_4T_5T_1(T_4^2 + T_5^2 + T_1^2) \\ &+ T_6T_1T_3(T_6^2 + T_1^2 + T_3^2) + T_2T_4T_6(T_2^2 + T_4^2 + T_6^2) \\ &= k^2T_1T_2T_3T_4T_5T_6(T_1T_4T_6 + T_3T_6T_2 + T_5T_2T_4 + T_1T_3T_5) \end{aligned} \quad \ldots\ldots\ldots\ldots (13),$$

$$\begin{aligned} &T_1T_4T_6(T_2^2 + T_3^2 + T_5^2) + T_3T_6T_2(T_4^2 + T_5^2 + T_1^2) \\ &+ T_5T_2T_4(T_6^2 + T_1^2 + T_3^2) + T_1T_3T_5(T_2^2 + T_4^2 + T_6^2) = 0 \ldots (14), \end{aligned}$$

$$\frac{\Sigma\{T_2T_3T_5(T_2^2 + T_3^2 + T_5^2)\}}{\Sigma\{T_1T_4T_6(T_1^2 + T_4^2 + T_6^2)\}} = k^2\frac{T_1T_2T_3T_4T_5T_6}{T_1^2 + T_2^2 + T_3^2 + T_4^2 + T_5^2 + T_6^2} \quad \ldots\ldots\ldots\ldots (15).$$

§ 5. In a similar manner, in virtue of the formula

$$\mathrm{sn}\left(k'u, \frac{ik}{k'}\right) = k'\frac{\mathrm{sn}\,u}{\mathrm{dn}\,u},$$

we may replace S_1, S_2, ...S_6 by $k'Q_1$, $k'Q_2$, ...$k'Q_6$ where Q_1, Q_2, &c. denote $\frac{\mathrm{sn}\,A}{\mathrm{dn}\,A}$, $\frac{\mathrm{sn}\,(B-C)}{\mathrm{dn}\,(B-C)}$, &c. if we also replace k^2 by $-\frac{k^2}{k'^2}$. We thus obtain from (2), (3), (4), (8), (9) five equations analogous to (11), ...(15). It is unnecessary to write these down as they differ from (11)...(15) only by the substitution of Q_1, Q_2, &c. for T_1, T_2, &c., and of $-k^2k'^2$ for k'^2.

The equations (2), (3), (11), (12), and the corresponding pair involving the Q's are well known; they are due to Gudermann and were published by him in Crelle's *Journal*, vol. xviii p. 167, 168.

§ 6. If we put $k=0$ so that the sn's and cn's become sines and cosines, then denoting $\sin A$, $\sin(B-C)$, &c. by

s_1, s_2, &c. and $\tan A$, $\tan(B-C)$, &c. by t_1, t_2, &c. we deduce from (2), (11), (3), (12), the equations

$$s_2s_3s_5 + s_4s_5s_1 + s_6s_1s_3 + s_2s_4s_6 = 0 \quad \ldots\ldots\ldots\ldots (16),$$

$$t_2t_3t_5 + t_4t_5t_1 + t_6t_1t_3 + t_2t_4t_6 = 0 \quad \ldots\ldots\ldots\ldots (17),$$

$$s_1s_2 + s_3s_4 + s_5s_6 = 0 \quad \ldots\ldots\ldots\ldots (18),$$

$$t_1t_2 + t_3t_4 + t_5t_6 + t_1t_2t_3t_4t_5t_6 = 0 \quad \ldots\ldots\ldots\ldots (19),$$

which are well-known results. From (4), (13), (8), (14), (15), we deduce that

$$\begin{aligned} &s_2s_3s_5\,(s_2^2+s_3^2+s_5^2) + s_4s_5s_1\,(s_4^2+s_5^2+s_1^2) \\ &+ s_6s_1s_3\,(s_6^2+s_1^2+s_3^2) + s_2s_4s_6\,(s_2^2+s_4^2+s_6^2) = 0 \quad \ldots\ldots\ldots\ldots (20), \\ &t_2t_3t_5\,(t_2^2+t_3^2+t_5^2) + t_4t_5t_1\,(t_4^2+t_5^2+t_1^2) \\ &+ t_6t_1t_3\,(t_6^2+t_1^2+t_3^2) + t_2t_4t_6\,(t_2^2+t_4^2+t_6^2) \\ &\qquad = t_1t_2t_3t_4t_5t_6\,(t_1t_4t_6 + t_3t_6t_2 + t_5t_2t_4 + t_1t_3t_5)) = 0 \ldots (21), \\ &s_1s_4s_6\,(s_2^2+s_3^2+s_5^2) + s_3s_6s_2\,(s_4^2+s_5^2+s_1^2) \\ &+ s_5s_2s_4\,(s_6^2+s_1^2+s_3^2) + s_1s_3s_5\,(s_2^2+s_4^2+s_6^2) = 0 \quad \ldots\ldots\ldots\ldots (22), \\ &t_1t_4t_6\,(t_2^2+t_3^2+t_5^2) + t_3t_6t_2\,(t_4^2+t_5^2+t_1^2) \\ &+ t_5t_2t_4\,(t_6^2+t_1^2+t_3^2) + t_1t_3t_5\,(t_2^2+t_4^2+t_6^2) = 0 \quad \ldots\ldots\ldots\ldots (23), \end{aligned}$$

$$\frac{\Sigma\{t_2t_3t_5\,(t_2^2+t_3^2+t_5^2)\}}{\Sigma\{t_1t_4t_6\,(t_1^2+t_4^2+t_6^2)\}} = \frac{t_1t_2t_3t_4t_5t_6}{t_1^2+t_2^2+t_3^2+t_4^2+t_5^2+t_6^2} \ldots (24),$$

§ 7. Expanding $\operatorname{sn}u$, $\operatorname{cn}u$, $\operatorname{dn}u$ in powers of k^2, it can be shown that

$$\left.\begin{aligned} \operatorname{sn}u &= \sin u - \tfrac{1}{4}k^2u\cos u + \tfrac{1}{8}k^2\sin 2u\cos u, \\ \operatorname{cn}u &= \cos u + \tfrac{1}{4}k^2u\sin u - \tfrac{1}{8}k^2\sin 2u\sin u, \\ \operatorname{dn}u &= 1 - \tfrac{1}{2}k^2\sin^2 u, \end{aligned}\right\}$$

and by substituting these values of $\operatorname{sn}u$, $\operatorname{cn}u$, $\operatorname{dn}u$ in formulæ involving elliptic functions and equating the coefficients of k^2 we may deduce trigonometrical equations which, are generally of more interest than those obtained by merely putting $k=0$. The terms $-\frac{1}{4}k^2u\cos u$ and $\frac{1}{4}k^2u\sin u$ in which the argument appears as an external factor will in general lead to systems of terms which are to be separately equated to zero, so that in elliptic function formulæ, such as those given in this paper, we may put

$$\left.\begin{aligned} \operatorname{sn}u &= \sin u\,(1 + \tfrac{1}{4}k^2\cos^2 u), \\ \operatorname{cn}u &= \cos u\,(1 - \tfrac{1}{4}k^2\sin^2 u), \\ \operatorname{dn}u &= \qquad 1 - \tfrac{1}{2}k^2\sin^2 u \end{aligned}\right\}$$

and equate to zero the coefficient of k^2. We thus deduce (20) from (2); and it was in this manner that I first obtained the equation (20), which was stated without proof in vol. IX. p. 124, and suggested to Mr. Wilkinson his theorem (1).

§ 8. I may observe that (20) may be readily verified, for on transforming the terms in brackets by means of the formula

$$\sin^2 x + \sin^2 y + \sin^2 (x-y) = 2 - 2\cos x \cos y \cos (x-y),$$

and denoting $\cos A$, $\cos(B-C)$ &c. by c_1, c_2, &c., it becomes

$$s_2 s_3 s_5 (1 - c_2 c_3 c_5) + s_4 s_5 s_1 (1 - c_4 c_5 c_1)$$
$$+ s_6 s_1 s_2 (1 - c_6 c_1 c_2) + s_2 s_4 s_6 (1 - c_2 c_4 c_6) = 0,$$

that is

$$s_2 s_3 s_5 \quad + s_4 s_5 s_1 \quad + s_6 s_1 s_2 \quad + s_2 s_4 s_6$$
$$- \tfrac{1}{8} (s'_2 s'_3 s'_5 + s'_4 s'_5 s'_1 + s'_6 s'_1 s'_2 + s'_2 s'_4 s'_6) = 0,$$

where s'_1, s'_2, &c. denote $\sin 2A$, $\sin(2B-2C)$ &c.; and each line of this equation is separately equal to zero in virtue of (16).

§ 9. By substituting in (3) for the sn's, as explained in § 7, and equating to zero the coefficient of k^2, we obtain the trigonometrical equation

$$s_1 s_2 (s_1^2 + s_2^2) + s_3 s_4 (s_3^2 + s_4^2) + s_5 s_6 (s_5^2 + s_6^2) - 4 s_1 s_2 s_3 s_4 s_5 s_6 = 0 \ldots (25),$$

and in the same manner we obtain from (4), after some reductions and transformations, the equation

$$s_2 s_3 s_5 (c_2^4 + c_3^4 + c_5^4 + 2c_2^2 c_3^2 c_5^2)$$
$$+ s_4 s_5 s_1 (c_4^4 + c_5^4 + c_1^4 + 2c_4^2 c_5^2 c_1^2)$$
$$+ s_6 s_1 s_2 (c_6^4 + c_1^4 + c_2^4 + 2c_6^2 c_1^2 c_2^2)$$
$$+ s_2 s_4 s_6 (c_2^4 + c_4^4 + c_6^4 + 2c_2^2 c_4^2 c_6^2)$$
$$+ 2 s_1 s_2 s_3 s_4 s_5 s_6 (s_1 s_4 s_6 + s_3 s_6 s_2 + s_5 s_2 s_4 + s_1 s_3 s_5) = 0.$$

ON A CERTAIN DEFINITE INTEGRAL.

By *Sir James Cockle, M.A., F.R.S.*

1. The following statement is a particular case of a theorem of Boole (*Phil. Trans.*, 1864, see pp. 752, 3); viz. if

$$\sqrt{(-1)} = i, \quad e^{-\frac{i\pi}{n}} = a, \quad V = \frac{v^{n-1}}{v^n + 1},$$

$$P = \frac{dV}{dv} = \frac{(n-1)\,v^{n-2} - v^{2n-2}}{(v^n+1)^2},$$

then, y being that real root of $y^n - xy^{n-1} - 1 = 0$ which reduces to 1 when $x = 0$, we have

$$y = 1 + \frac{1}{2i\pi}\left(\int_0^\infty \frac{xvPdv}{1 - axV} - \int_0^\infty \frac{xvPdv}{1 - \frac{x}{a}V}\right).$$

2. Put, for the purpose of evaluation,

$$v^n + 1 = u, \quad v^n - axv^{n-1} + 1 = U, \quad \frac{xvP}{1 - axV} = \phi;$$

then

$$\phi = -\frac{n}{a}\frac{1}{u} + \frac{-xv^{n-1} + \frac{n}{a}}{U},$$

$$\frac{d\phi}{dx} = \frac{-v^{2n-1} + (n-1)\,v^{n-1}}{U^2}$$

$$\frac{d\phi}{dv} = \frac{n^2}{a}\frac{v^{n-1}}{u^2} + \frac{xv^{2n-2} - \frac{n^2}{a}v^n + (n-1)^2 xv^{n-1}}{U^2},$$

and, since we may put

$$\phi = -\frac{n}{a}\frac{u}{u^2} + \frac{-xv^{2n-1} + ax^2v^{2n-2} + \frac{n}{a}v^n - (n+1)\,xv^{n-1} + \frac{n}{a}}{U^2},$$

consequently

$$\frac{d\phi}{dx} + \zeta\phi + \varepsilon v\frac{d\phi}{dv} = \frac{b_0 v^n + b_n}{u^2} + U^{-2}\left(B_0 v^{2n-1} + B_1 v^{2n-2} + B_{n-1}v^n + B_n v^{n-1} + B_{2n-1}\right),$$

wherein

$$b_0=\frac{n}{a}(n\varepsilon-\zeta),\quad b_n=-\frac{n}{a},$$

$$B_0=x(\varepsilon-\zeta)-1,\quad B_1=ax^2\zeta,\quad B_{n-1}=\frac{n}{a}(\zeta-n\varepsilon),$$

$$B_n=n-1-(n+1)\,x\zeta+(n-1)^2x\varepsilon,\quad B_{2n-1}=\frac{n}{a}\zeta.$$

3. Consider the case of $n=2$. Here $B_1v^{2n-2}+B_{n-1}v^n$ becomes $\{ax^2\zeta-\frac{2}{a}(2\varepsilon-\zeta)\}v^2$, and we have, changing the form of our expression for a reason that will forthwith appear,

$$\frac{d\phi}{dx}+\zeta\phi+\varepsilon v\frac{d\phi}{dv}=\frac{2}{a}\cdot\frac{(2\varepsilon-\zeta)v^2-\zeta}{(v^2+1)^2}+\chi(A,B),$$

where $$\chi(A,B)=\frac{B_0v^3+B_1v^2+B_2v+B_3}{(A_0v^2+A_1v+A_2)^2},$$

and $$A_0=1,\quad A_1=-ax,\quad A_2=1,$$

$$B_0=x(\varepsilon-\zeta-1,)\quad B_1=ax^2\zeta-\frac{2}{a}(2\varepsilon-\zeta),$$

$$B_2=x(\varepsilon-3\zeta)+1,\quad B_3=\frac{2}{a}\zeta.$$

4. Let ζ and ε be functions of x only; integrate with respect to v; then

$$\frac{d}{dx}\int\phi dv+(\zeta-\varepsilon)\int\phi dv+\varepsilon v\phi$$

$$=\frac{2}{a}\left\{(\varepsilon-\zeta)\arctan v-\varepsilon\frac{v}{v^2+1}\right\}+\int\chi(A,B)\,dv.$$

5. Our object is to find $\int\chi(A,B)\,dv$ without decomposing its denominator. Such decomposition would amount to assuming a knowledge of the Algebraical Solution of the quadratic, while the only solution at present to be regarded as known is that afforded by the definite integral. But I have shown in my paper "On Certain Rational Fractions" (Old Series, vol. IV., p. 168; No. XV., Jan. 1868), that if (see p. 175)

$$4A_0^3B_3-2A_0^2A_1B_2+4A_0^2A_2B_1-(6A_0A_1A_2-A_1^3)B_0=0,$$

the integration can be effected without the decomposition. Substituting for the A's and B's in this result, we obtain the condition

$$\zeta - \varepsilon = -\frac{a^2 x}{a^2 x^2 - 4}.$$

6. Taking the integral in Art. 4 between the limits 0 and ∞, we have, after evaluating the integrated parts,

$$\frac{d}{dx}\int_0^\infty \phi dv - \frac{a^2 x}{a^2 x^2 - 4}\int_0^\infty \phi dv - \varepsilon x = \pi \frac{ax}{a^2 x^2 - 4} + \int_0^\infty \chi(A, B) dv.$$

7. Now (see a solution "By R.", perhaps Rutherford, at p. 38 of No. I of vol. III. of *The Mathematician*, November, 1847), $e^{\frac{1}{2}(i\pi)} = i$ and $e^{-\frac{1}{2}(i\pi)} = -i$. Hence $a = -i$ and $\frac{1}{a} = i$.

8. Perchance ε might be so assigned as slightly to facilitate calculation. But I shall here take $\varepsilon = 0$. We shall then have

$$\int \chi(A, B)\, dv = \frac{2 \log U}{a^2 x^2 - 4} + \frac{axv - 1}{U} + C,$$

and, taking this between the limits and making the appropriate substitutions in Art. 6, we get

$$\frac{d}{dx}\int_0^\infty \phi dv - \frac{x}{x^2 + 4}\int_0^\infty \phi dv = \frac{i\pi x}{x^2 + 4} + 1 + \frac{2}{x^2 + 4}\left(\log U\right)_0^\infty.$$

9. The change of a into $\frac{1}{a}$ (*i.e.* that of i into $-i$) gives rise to corresponding formulæ. I denote the transition by a suffixed 2. We thus get

$$\frac{d}{dx}\int_0^\infty \phi_2 dv - \frac{x}{x^2+4}\int_0^\infty \phi_2 dv = -\frac{i\pi x}{x^2+4} + 1 + \frac{2}{x^2+4}\left(\log U_2\right)_0^\infty.$$

and, subtracting from the result of Art. 8,

$$\frac{d}{dx}\int_0^\infty (\phi - \phi_2)\, dv - \frac{x}{x^2+4}\int_0^\infty (\phi - \phi_2)\, dv = \frac{2i\pi x}{x^2+4} + \frac{2}{x^2+4}\left(\log \frac{U}{U_2}\right)_0^\infty,$$

whence

$$\int_0^\infty (\phi - \phi_2)\, dv = -2i\pi + \tfrac{1}{2}x \left(\log \frac{U}{U_2}\right)_0^\infty + c\sqrt{(x^2+4)},$$

c being an arbitrary constant introduced by the x-integration. Therefore

$$y = 1 + \frac{1}{2i\pi}\int_0^\infty (\phi - \phi_2)\, dv = \frac{1}{2i\pi}\left\{\tfrac{1}{2}x \left(\log \frac{U}{U_2}\right)_0^\infty + c\sqrt{(x^2+4)}\right\},$$

10. Put $\frac{1}{2i\pi}\left(\log \frac{U}{U_2}\right)_0^\infty = M$, and substitute the resulting value of y in the quadratic. We have, inasmuch as x is to remain undetermined,

$$c^2 = i^2\pi^2, \quad M = 1, \quad y = \tfrac{1}{2}x \pm \sqrt{(1 + \tfrac{1}{4}x^2)},$$

which last expression is right.

11. Since

$$M = \frac{1}{2i\pi}\left(\log \frac{U}{U_2}\right)_0^\infty$$

$$= \frac{1}{2i\pi}(\log 1 - \log 1) = \frac{1}{2i\pi}(2i\pi m - 2i\pi r) = m - r,$$

m and r being any whole numbers, consequently the determination $M = 1$ is legitimate.

We might of course have used Graves's form for the logarithm instead of Euler's.

2, Sandringham Gardens,
Ealing, near London, W.,
October 7, 1880.

ERRATUM.

Vol. x. (N. S.) foot of p. 16 *for* $\sqrt{\{(2x) - 1\}}$ *read* $\sqrt{(2x) - 1}$.

NOTE ON ELLIPTIC FUNCTIONS.

By *N. M. Ferrers.*

THE following formulæ were suggested by Mr. Glaisher's paper in the *Quarterly Journal*, vol. XVI., p. 382.

Putting x for $\operatorname{sn} u$, y for $\operatorname{sn} v$, we know that

$$\operatorname{sn}^2(u+v)\operatorname{sn}(u-v)=\frac{x^2-y^2}{1-k^2x^2y^2};$$

therefore

$$\operatorname{sn}(u+v)\operatorname{sn}(u-v)+A=\frac{A+x^2-y^2-Ak^2x^2y^2}{1-k^2x^2y^2}.$$

If the numerator of this fraction break up into factors

$$\begin{vmatrix} 2A, & -1, & 1 \\ -1, & 0, & -Ak^2 \\ 1, & -Ak^2, & 0 \end{vmatrix}=0,$$

therefore $$-2A^2k^2+2Ak^2=0;$$

therefore $$A=\pm\frac{1}{k}.$$

Therefore

$$k\operatorname{sn}(u+v)\operatorname{sn}(u-v)\pm 1=\pm\frac{(1\pm kx^2)(1\mp ky^2)}{1-k^2x^2y^2}\quad\ldots(1),$$

the upper signs in the ambiguities in all cases going together.

Again

$$\operatorname{cn}(u+v)\operatorname{cn}(u-v)=\frac{(1-x^2)(1-y^2)-x^2y^2(1-k^2x^2)(1-k^2y^2)}{(1-k^2x^2y^2)^2}$$

$$=\frac{1-x^2-y^2+k^2x^2y^2}{1-k^2x^2y^2};$$

therefore

$$\operatorname{cn}(u+v)\operatorname{cn}(u-v)+B=\frac{1+B-x^2-y^2+k^2(1-B)x^2y^2}{1-k^2x^2y^2}.$$

If the numerator of this fraction break up into factors

$$\begin{vmatrix} 2(1+B), & -1, & -1 \\ -1, & 0, & k^2(1-B) \\ -1, & k^2(1-B), & 0 \end{vmatrix}=0;$$

therefore $$2k^2(1-B^2)(1-B)+2k^2(1-B)=0,$$

therefore $$B^2=1-\frac{1}{k^2}=-\frac{k'^2}{k^2},$$

and $$B=\pm\frac{ik'}{k};$$

therefore

$$k\,\mathrm{cn}\,(u+v)\,\mathrm{cn}\,(u-v)\pm ik' = \frac{k\pm ik'-k\,(x^2+y^2)+k^2\,(k\mp ik')\,x^2y^2}{1-k^2x^2y^2}$$

$$= \frac{(k\pm ik')^2-k(k\pm ik')\,(x^2+y^2)+k^2x^2y^2}{(k\pm ik')\,(1-k^2x^2y^2)}$$

$$= \frac{(k\pm ik'-kx^2)\,(k\pm ik'-ky^2)}{(k\pm ik')\,(1-k^2x^2y^2)} \quad \ldots(2),$$

the upper signs, as before, going together.

Lastly

$$\mathrm{dn}\,(u+v)\,\mathrm{dn}\,(u-v) = \frac{(1-k^2x^2)\,(1-k^2y^2)-\eta^2x^2y^2\,(1-x^2)\,(ky^2)}{(1-k^2x^2y^2)^2}$$

$$= \frac{1-k^2\,(x^2+y^2)+k^2x^2y^2}{1-k^2x^2y^2};$$

therefore

$$\mathrm{dn}\,(u+v)\,\mathrm{dn}\,(u-v)+C = \frac{1+C-k^2\,(x^2+y^2)+k^2\,(1-C)\,x^2y^2}{1-k^2x^2y^2}.$$

If the numerator of this fraction break up into factors,

$$\begin{vmatrix} 2\,(1+C), & -k^2 & , & -k^2 \\ -k^2 & , & 0 & , & k^2\,(1-C) \\ -k^2 & , & k^2\,(1-C), & 0 & , \end{vmatrix} = 0;$$

therefore $\quad -2\,(1+C)\,(1-C)^2+2k^2\,(1-C)=0;$

therefore $\quad 1-C^2 = k^2,$

$$C = \pm k';$$

therefore

$$\mathrm{dn}\,(u+v)\,\mathrm{dn}\,(u-v)\pm k' = \frac{1\pm k'-k^2\,(x^2+y^2)+k^2\,(1\mp k')\,x^2y^2}{1-k^2x^2y^2}$$

$$= \frac{(1\pm k'-k^2x^2)\,(1\pm k'-k^2y^2)}{(1\pm k')\,(1-k^2x^2y^2)} \quad \ldots.(3).$$

We may write these formulæ

$$\mathrm{sn}\,(u+v)\,\mathrm{sn}\,(u-v)\pm\frac{1}{k} = \pm\frac{(1\pm k\,\mathrm{sn}^2u)\,(1\mp k\,\mathrm{sn}^2v)}{1-k^2\,\mathrm{sn}^2u\,\mathrm{sn}^2v},$$

$$k\,\mathrm{cn}\,(u+v)\,\mathrm{cn}\,(u-v)\pm ik' = \frac{(k\pm ik'-k\,\mathrm{sn}^2u)\,(k\pm ik'-k\,\mathrm{sn}^2v)}{(k\pm ik')\,(1-k^2\,\mathrm{sn}^2u\,\mathrm{sn}^2v)},$$

$$\mathrm{dn}\,(u+v)\,\mathrm{dn}\,(u-v)\pm k' = \frac{(1\pm k'-k^2\,\mathrm{sn}^2u)\,(1\pm k'-k^2\,\mathrm{sn}^2v)}{(1\pm k')\,(1-k^2\,\mathrm{sn}^2u\,\mathrm{sn}^2v)},$$

or, finally,

$$k\,\mathrm{cn}\,(u+v)\,\mathrm{cn}\,(u-v)\pm ik' = \frac{(k\,\mathrm{cn}^2u\pm ik')\,(k\,\mathrm{cn}^2v\pm ik')}{(k\pm ik')\,(1-k^2\,\mathrm{sn}^2u\,\mathrm{sn}^2v)},$$

$$\mathrm{dn}\,(u+v\,\mathrm{dn}\,(u-v)\pm k' = \frac{(\mathrm{dn}^2u\pm k')\,(\mathrm{dn}^2v\pm k')}{(1\pm k')\,(1-k^2\,\mathrm{sn}^2u\,\mathrm{sn}^2v)}.$$

SYSTEMS OF FORMULÆ IN ELLIPTIC FUNCTIONS.

By *J. W. L. Glaisher.*

THE formulæ contained in this paper, which were calculated for my lectures on Elliptic Functions, form the complete systems of equations corresponding to several of the groups given in Chapter IV. of Cayley's *Treatise on Elliptic Functions* (1876), and Gudermann's *Theorie der Modular-Functionen und der Modular-Integrale* (Crelle's Journal, t. XVIII.).

§§ 1, 2. *Formulæ for* sn^2u, cn^2u, dn^2u *in terms of* $\mathrm{cn}2u$ *and* $\mathrm{dn}2u$.

§ 1. From the fundamental formulæ

$$\mathrm{cn}\,2u = \frac{1 - 2\,\mathrm{sn}^2u + k^2\,\mathrm{sn}^4u}{1 - k^2\,\mathrm{sn}^4u},$$

$$\mathrm{dn}\,2u = \frac{1 - 2k^2\,\mathrm{sn}^2u + k^2\,\mathrm{sn}^4u}{1 - k^2\,\mathrm{sn}^4u},$$

we deduce the system of equations

$$1 + \mathrm{cn}\,2u = \frac{2\mathrm{cn}^2u}{1 - k^2\,\mathrm{sn}^4u},$$

$$1 - \mathrm{cn}\,2u = \frac{2\,\mathrm{sn}^2u\,\mathrm{dn}^2u}{1 - k^2\,\mathrm{sn}^4u},$$

$$1 + \mathrm{dn}\,2u = \frac{2\mathrm{dn}^2u}{1 - k^2\,\mathrm{sn}^4u},$$

$$1 - \mathrm{dn}\,2u = \frac{2k^2\,\mathrm{sn}^2u\,\mathrm{cn}^2u}{1 - k^2\,\mathrm{sn}^4u},$$

$$\mathrm{dn}\,2u + \mathrm{cn}\,2u = \frac{2\,\mathrm{cn}^2u\,\mathrm{dn}^2u}{1 - k^2\,\mathrm{sn}^4u},$$

$$\mathrm{dn}\,2u - \mathrm{cn}\,2u = \frac{2k'^2\,\mathrm{sn}^2u}{1 - k^2\,\mathrm{sn}^4u},$$

$$k'^2 + k^2\,\mathrm{cn}\,2u + \mathrm{dn}\,2u = \frac{2\,\mathrm{dn}^2u}{1 - k^2\,\mathrm{sn}^4u},$$

$$-k'^2 + k^2\,\mathrm{cn}\,2u + \mathrm{dn}\,2u = \frac{2k^2\,\mathrm{cn}^4u}{1 - k^2\,\mathrm{sn}^4u},$$

$$k'^2 - k^2\,\mathrm{cn}\,2u + \mathrm{dn}\,2u = \frac{2k'^2}{1 - k^2\,\mathrm{sn}^4u},$$

$$k'^2 + k^2\,\mathrm{cn}\,2u - \mathrm{dn}\,2u = -\frac{2k^2k'^2\,\mathrm{sn}^4u}{1 - k^2\,\mathrm{sn}^4u},$$

The numerators in these ten formulæ involve the squares of $\mathrm{sn}\,u$, $\mathrm{cn}\,u$, $\mathrm{dn}\,u$, their fourth powers and the products of their squares in pairs, one of them being independent of u.

§ 2. The formulæ in the last section lead at once to the following system of values of sn^2u, cn^2u, dn^2u,

$$\mathrm{sn}^2u = \frac{1-C}{1+D} = \frac{1}{k^2}\frac{1-D}{1+C} = -\frac{1}{k^2}\frac{k'^2+k^2C-D}{D-C} = \frac{D-C}{k'^2-k^2C+D},$$

$$\mathrm{cn}^2u = \frac{D+C}{1+D} = \frac{1}{k^2}\frac{-k'^2+k^2C+D}{1+C} = \frac{k'^2}{k^2}\frac{1-D}{D-C} = k'^2\frac{1+C}{k'^2-k^2C+D},$$

$$\mathrm{dn}^2u = \frac{k'^2+k^2C+D}{1+D} = \frac{D+C}{1+C} = k'^2\frac{1-C}{D-C} = k'^2\frac{1+D}{k'^2-k^2C+D},$$

where C, D denote $\mathrm{cn}\,2u$, $\mathrm{dn}\,2u$ respectively.

The formulæ in these two sections are the systems which include those given by Cayley in Arts. 100, 101, and by Gudermann in *Crelle*, t. xviii. pp. 145, 146.

§ 3. *Formulæ for* $\mathrm{sn}\,(u+v)$, $\mathrm{cn}\,(u+v)$, $\mathrm{dn}\,(u+v)$.

§ 3. This system is given by Cayley in Art. 91, but in an algebraical form.* Expressed in the elliptic function notation, the system becomes

* The corresponding system of formulæ which are integrals of the differential equation

$$\frac{dx}{\sqrt{\{(a-x)(b-x)(c-x)(d-x)\}}} + \frac{dy}{\sqrt{\{(a-y)(b-y)(c-y)(d-y)\}}} = 0$$

have since been published by Cayley in the *Messenger* for August, 1878 (vol. VIII., pp. 60–62).

$$\operatorname{sn}(u+v) = \frac{s_1c_2d_2 + s_2c_1d_1}{1-k^2s_1^2s_2^2} = \frac{s_1^2 - s_2^2}{s_1c_2d_2 - s_2c_1d_1} = \frac{s_1c_1d_2 + s_2c_2d_1}{c_1c_2 + s_1s_2d_1d_2} = \frac{s_1d_1c_2 + s_2d_2c_1}{d_1d_2 + k^2s_1s_2c_1c_2},$$

$$\operatorname{cn}(u+v) = \frac{c_1c_2 - s_1s_2d_1d_2}{1-k^2s_1^2s_2^2} = \frac{s_1c_1d_2 - s_2c_2d_1}{s_1c_2d_2 - s_2c_1d_1} = \frac{1-s_1^2-s_2^2+k^2s_1^2s_2^2}{c_1c_2 + s_1s_2d_1d_2} = \frac{c_1c_2d_1d_2 - k'^2s_1s_2}{d_1d_2 + k^2s_1s_2c_1c_2},$$

$$\operatorname{dn}(u+v) = \frac{d_1d_2 - k^2s_1s_2c_1c_2}{1-k^2s_1^2s_2^2} = \frac{s_1d_1c_2 - s_2d_2c_1}{s_1d_2c_2 - s_2c_1d_1} = \frac{c_1c_2d_1d_2 + k'^2s_1s_2}{c_1c_2 + s_1s_2d_1d_2} = \frac{1-k^2s_1^2-k^2s_2^2+k^2s_1^2s_2^2}{d_1d_2 + k^2s_1s_2c_1c_2},$$

where s_1, c_1, d_1, s_2, c_2, d_2 denote $\operatorname{sn}u$, $\operatorname{cn}u$, $\operatorname{dn}u$, $\operatorname{su}v$, $\operatorname{cn}v$, $\operatorname{dn}v$ respectively.

§§ 4, 5. *Values of* $\operatorname{sn}(u+\frac{1}{2}K)$, *&c. in terms of* $\operatorname{sn}u$, $\operatorname{cn}u$, $\operatorname{dn}u$.

§ 4. By putting $v = \frac{1}{2}K$, $\frac{1}{2}iK'$, $\frac{1}{2}K + \frac{1}{2}iK'$ in the formulæ of the last section, and reducing, we obtain the following system of formulæ giving $\operatorname{sn}^2(u+\frac{1}{2}K)$, &c. in terms of $\operatorname{sn}u$, $\operatorname{cn}u$, $\operatorname{dn}u$,

$$\frac{\operatorname{sn}(u+\frac{1}{2}K)}{\operatorname{sn}\frac{1}{2}K} = \frac{cd+k's}{1-(1-k')s^2} = \frac{1-(1+k')s^2}{cd-k's} = \frac{d+(1+k')sc}{c+sd} = \frac{c+sd}{d+(1-k')sc},$$

$$\frac{\operatorname{cn}(u+\frac{1}{2}K)}{\operatorname{cn}\frac{1}{2}K} = \frac{c-sd}{1-(1-k')s^2} = \frac{d-(1+k')sc}{cd-k's} = \frac{1-(1+k')s^2}{c+sd} = \frac{cd-k's}{d+(1-k')sc},$$

$$\frac{\operatorname{dn}(u+\frac{1}{2}K)}{\operatorname{dn}\frac{1}{2}K} = \frac{d-(1-k')sc}{1-(1-k')s^2} = \frac{c-sd}{cd-k's} = \frac{cd+k's}{c+sd} = \frac{1-(1-k')s^2}{d+(1-k')sc},$$

$$\frac{\operatorname{sn}(u+\frac{1}{2}iK')}{\operatorname{sn}\frac{1}{2}iK'} = \frac{cd-i(1+k)s}{1+ks^2} = \frac{1+ks^2}{cd+i(1+k)s} = \frac{d-iksc}{c+isd} = \frac{c-isc}{d+iksc},$$

$$\frac{\operatorname{cn}(u+\frac{1}{2}iK')}{\operatorname{cn}\frac{1}{2}iK'} = \frac{c-isd}{1+ks^2} = \frac{d+iksc}{cd+i(1+k)s} = \frac{1-ks^2}{c+isd} = \frac{cd-i(1-k)s}{d+iksc},$$

$$\frac{\mathrm{dn}\,(u+\frac{1}{2}iK')}{\mathrm{dn}\,\frac{1}{2}iK'} = \frac{d-iksc}{1+ks^2} = \frac{c+isd}{cd+i(1+k)s} = \frac{cd+i(1-k)s}{c+isd} = \frac{1-ks^2}{d+iksc},$$

$$\frac{\mathrm{sn}\,(u+\frac{1}{2}K+\frac{1}{2}iK')}{\mathrm{sn}\,(\frac{1}{2}K+\frac{1}{2}iK')} = \frac{cd-ik's}{1-k(k+ik')s^2} = \frac{1-k(k-ik')s^2}{cd+ik's} = \frac{d+ksc}{c+(k+ik')sd} = \frac{c+(k-ik')sd}{d+ksc},$$

$$\frac{\mathrm{cn}\,(u+\frac{1}{2}K+\frac{1}{2}iK')}{\mathrm{cn}\,(\frac{1}{2}K+\frac{1}{2}iK)} = \frac{c-(k+ik')sd}{1-k(k+ik')s^2} = \frac{d-ksc}{cd+ik's} = \frac{1-k(k+ik')s^2}{c+(k+ik')sd} = \frac{cd-ik's}{d+ksc},$$

$$\frac{\mathrm{dn}\,(u+\frac{1}{2}K+\frac{1}{2}iK')}{\mathrm{dn}\,(\frac{1}{2}K+\frac{1}{2}iK')} = \frac{d-ksc}{1-k(k+ik')s^2} = \frac{c-(k-ik')sd}{cd+ik's} = \frac{cd+ik's}{c+(k+ik')sd} = \frac{1-k(k-ik')s^2}{d+ksc},$$

where s, c, d denote $\mathrm{sn}\,u$, $\mathrm{cn}\,u$, $\mathrm{dn}\,u$ respectively, and

$$\mathrm{sn}\,\tfrac{1}{2}K = \frac{1}{\sqrt{(1+k')}}, \quad \mathrm{sn}\,\tfrac{1}{2}iK' = \frac{i}{\sqrt{k}}, \quad \mathrm{sn}\,(\tfrac{1}{2}K+\tfrac{1}{2}iK') = \frac{1}{\sqrt{2}}\,\frac{\sqrt{(1+k)}+i\sqrt{(1-k)}}{\sqrt{k}} = \frac{e^{\frac{1}{4}i\pi-\frac{1}{2}i\theta}}{\sqrt{k}},$$

$$\mathrm{cn}\,\tfrac{1}{2}K = \frac{\sqrt{k'}}{\sqrt{(1+k')}}, \quad \mathrm{cn}\,\tfrac{1}{2}iK' = \frac{\sqrt{(1+k)}}{\sqrt{k}}, \quad \mathrm{cn}\,(\tfrac{1}{2}K+\tfrac{1}{2}iK') = \frac{1-i}{\sqrt{2}}\,\frac{\sqrt{k'}}{\sqrt{k}} = \frac{\sqrt{k'}}{\sqrt{k}}\,e^{-\frac{1}{4}i\pi},$$

$$\mathrm{dn}\,\tfrac{1}{2}K = \sqrt{k'}, \quad \mathrm{dn}\,\tfrac{1}{2}iK' = \sqrt{(1+k)}, \quad \mathrm{dn}\,(\tfrac{1}{2}K+\tfrac{1}{2}iK') = \frac{\sqrt{k'}}{\sqrt{2}}\,\{\sqrt{(1+k')}-i\sqrt{(1-k')}\} = \sqrt{k'}.e^{-\frac{1}{2}i\theta},$$

θ being the modular angle.

§ 5. Since

$$\frac{\operatorname{sn}(u+\frac{1}{2}K)}{\operatorname{sn}\frac{1}{2}K}=\frac{d+(1+k')sc}{c+sd}=\frac{c+sd}{d+(1-k')sc},$$

we have, by multiplying these values,

$$\frac{\operatorname{sn}^2(u+\frac{1}{2}K)}{\operatorname{sn}^2\frac{1}{2}K}=\frac{d+(1+k')sc}{d+(1-k')sc},$$

and in this manner, by multiplying together pairs of values, the following formulæ were obtained:

$$\frac{\operatorname{sn}^2(u+\frac{1}{2}K)}{\operatorname{sn}^2\frac{1}{2}K}=\frac{d+(1+k')sc}{d+(1-k')sc},\quad \frac{\operatorname{sn}^2(u+\frac{1}{2}iK')}{\operatorname{sn}^2\frac{1}{2}iK'}=\frac{cd-i(1+k)s}{cd+i(1+k)s},\quad \frac{\operatorname{sn}^2(u+\frac{1}{2}K+\frac{1}{2}iK')}{\operatorname{sn}^2(\frac{1}{2}K+\frac{1}{2}iK')}=\frac{c+(k-ik')sd}{c+(k+ik')sd},$$

$$\frac{\operatorname{cn}^2(u+\frac{1}{2}K)}{\operatorname{cn}^2\frac{1}{2}K}=\frac{d-(1+k')sc}{d+(1-k')sc},\quad \frac{\operatorname{cn}^2(u+\frac{1}{2}iK')}{\operatorname{cn}^2\frac{1}{2}iK'}=\frac{cd-i(1-k)s}{cd+i(1+k)s},\quad \frac{\operatorname{cn}^2(u+\frac{1}{2}K+\frac{1}{2}iK')}{\operatorname{cn}^2(\frac{1}{2}K+\frac{1}{2}iK')}=\frac{c-(k+ik')sd}{c+(k+ik')sd},$$

$$\frac{\operatorname{dn}^2(u+\frac{1}{2}K)}{\operatorname{dn}^2\frac{1}{2}K}=\frac{d-(1-k')sc}{d+(1-k')sc},\quad \frac{\operatorname{dn}^2(u+\frac{1}{2}iK')}{\operatorname{dn}^2\frac{1}{2}iK'}=\frac{cd+i(1-k)s}{cd+i(1+k)s},\quad \frac{\operatorname{dn}^2(u+\frac{1}{2}K+\frac{1}{2}iK')}{\operatorname{dn}^2(\frac{1}{2}K+\frac{1}{2}iK')}=\frac{c-(k-ik')sd}{c+(k+ik')sd},$$

where s, c, d denote $\operatorname{sn}u$, $\operatorname{cn}u$, $\operatorname{dn}u$ respectively as before, and

$$\operatorname{sn}^2\tfrac{1}{2}K=\frac{1}{1+k'},\quad \operatorname{sn}^2\tfrac{1}{2}iK'=-\frac{1}{k},\quad \operatorname{sn}^2(\tfrac{1}{2}K+\tfrac{1}{2}iK')=\frac{k+ik'}{k},$$

$$\operatorname{cn}^2\tfrac{1}{2}K=\frac{k'}{1+k'},\quad \operatorname{cn}^2\tfrac{1}{2}iK'=\frac{1+k}{k},\quad \operatorname{cn}^2(\tfrac{1}{2}K+\tfrac{1}{2}iK')=-\frac{ik'}{k},$$

$$\operatorname{dn}^2\tfrac{1}{2}K=k',\quad \operatorname{dn}^2\tfrac{1}{2}iK'=1+k,\quad \operatorname{dn}^2(\tfrac{1}{2}K+\tfrac{1}{2}iK)=k'(k'-ik),$$

§§ 6, 7. *Values of* $\mathrm{sn}^2(u+\frac{1}{2}K)$, *&c. in terms of* $\mathrm{sn}\,2u$, $\mathrm{cn}\,2u$, $\mathrm{dn}\,2u$.

§ 6. Putting $u+\frac{1}{2}K$ in place of u in the formula

$$\mathrm{sn}^2 u = \frac{1-\mathrm{cn}\,2u}{1+\mathrm{dn}\,2u},$$

we have

$$\mathrm{sn}^2(u+\tfrac{1}{2}K) = \frac{1-\mathrm{cn}\,(2u+K)}{1+\mathrm{dn}\,(2u+K)} = \frac{\mathrm{dn}\,2u+k'\,\mathrm{sn}\,2u}{\mathrm{dn}\,2u+k'},$$

and, substituting in this manner in the system of formula in § 2, we obtain the following system, giving $\mathrm{sn}^2(u+\frac{1}{2}K)$, &c. in terms of $\mathrm{sn}\,2u$, $\mathrm{cn}\,2u$, $\mathrm{dn}\,2u$,

$$\mathrm{sn}^2(u+\tfrac{1}{2}K) = \frac{D+k'S}{D+k'} = \frac{1}{k^2}\,\frac{D-k'}{D-k'S} = \frac{1}{k^2}\,\frac{1+k^2S-k'D}{1+S} = \frac{1+S}{1+k^2S+k'D},$$

$$\mathrm{cn}^2(u+\tfrac{1}{2}K) = k'\,\frac{1-S}{D+k'} = \frac{k'}{k^2}\,\frac{1-k^2S-k'D}{D-k'S} = \frac{k'}{k^2}\,\frac{D-k'}{1+S} = k'\,\frac{D-k'S}{1+k^2S+k'D},$$

$$\mathrm{dn}^2(u+\tfrac{1}{2}K) = k'\,\frac{1-k^2S+k'D}{D+k'} = k'\,\frac{1-S}{D-k'S} = k'\,\frac{D+k'S}{1+S} = k'\,\frac{D+k'}{1+k^2S+k'D},$$

$$\operatorname{sn}^2(u+\tfrac{1}{2}iK') = -\frac{1}{k}\,\frac{D-ikS}{C+iS} = -\frac{1}{k}\,\frac{C-iS}{D+ikS} = -\frac{1}{k}\,\frac{C-kD-ik'^2S}{D-kC} = -\frac{1}{k}\,\frac{D-kC}{C-kD+ik'^2S},$$

$$\operatorname{cn}^2(u+\tfrac{1}{2}iK') = \frac{1}{k}\,\frac{D+kC}{C+iS} = \frac{1}{k}\,\frac{C+kD-ik'^2S}{D+ikS} = \frac{k'^2}{k}\,\frac{C-iS}{D-kC} = \frac{k'^2}{k}\,\frac{D+ikS}{C-kD+ik'^2S},$$

$$\operatorname{dn}^2(u+\tfrac{1}{2}iK') = \frac{C+kD+ik'^2S}{C+iS} = \frac{D+kC}{D+ikS} = k'^2\,\frac{D-ikS}{D-kC} = k'^2\,\frac{C+iS}{C-kD+ik'^2S},$$

$$\operatorname{sn}^2(u+\tfrac{1}{2}K+\tfrac{1}{2}iK') = \frac{1}{k}\,\frac{k'-ikC}{k'S-iC} = \frac{1}{k}\,\frac{k'S+iC}{k'+ikC} = \frac{1}{k}\,\frac{k+S+ik'C}{1+kS} = \frac{1}{k}\,\frac{1+kS}{k+S-ik'C},$$

$$\operatorname{cn}^2(u+\tfrac{1}{2}K+\tfrac{1}{2}iK') = -\frac{k'}{k}\,\frac{1-kS}{k'S-iC} = \frac{k'}{k}\,\frac{k-S-ik'C}{k'+ikC} = -\frac{k'}{k}\,\frac{k'S+iC}{1+kS} = -\frac{k'}{k}\,\frac{k'+ikC}{k+S-ik'C},$$

$$\operatorname{dn}^2(u+\tfrac{1}{2}K+\tfrac{1}{2}iK') = -k'\,\frac{k-S+ik'C}{k'S-iC} = k'\,\frac{1-kS}{k'+ikC} = k'\,\frac{k'-ikC}{1+kS} = k'\,\frac{k'S-iC}{k+S-ik'C},$$

where S, C, D denote $\operatorname{sn}2u$, $\operatorname{cn}2u$, $\operatorname{dn}2u$, respectively.

The formulæ given by Cayley in Arts. 103–106 belong to the systems contained in §§ 4–6.

§ 7. By multiplying together pairs of values, as in § 5, in the formulæ of §§ 2 and 6 we deduce the system

$$\mathrm{sn}^4 u = -\frac{1}{k^2}\frac{k'^2+k^2C-D}{k'^2-k^2C+D},$$

$$\mathrm{cn}^4 u = \frac{k'^2}{k^2}\frac{-k'^2+k^2C+D}{k'^2-k^2C+D},$$

$$\mathrm{dn}^4 u = k'^2\frac{k'^2+k^2C+D}{k'^2-k^2C+D},$$

$$\mathrm{sn}^4(u+\tfrac{1}{2}K) = \frac{1}{k^2}\frac{1+k^2S-k'D}{1+k^2S+k'D},$$

$$\mathrm{cn}^4(u+\tfrac{1}{2}K) = \frac{k'^2}{k^2}\frac{1-k^2S-k'D}{1+k^2S+k'D},$$

$$\mathrm{dn}^4(u+\tfrac{1}{2}K) = k'^2\frac{1-k^2S+k'D}{1+k^2S+k'D},$$

$$\mathrm{sn}^4(u+\tfrac{1}{2}iK') = \frac{1}{k^2}\frac{C-kD-ik'^2S}{C-kD+ik'^2S},$$

$$\mathrm{cn}^4(u+\tfrac{1}{2}iK') = \frac{k'^2}{k^2}\frac{C+kD-ik'^2S}{C-kD+ik'^2S},$$

$$\mathrm{dn}^4(u+\tfrac{1}{2}iK') = k'^2\frac{C+kD+ik'^2S}{C-kD+ik'^2S},$$

$$\mathrm{sn}^4(u+\tfrac{1}{2}K+\tfrac{1}{2}iK') = \frac{1}{k^2}\frac{k+S+ik'C}{k+S-ik'C},$$

$$\mathrm{cn}^4(u+\tfrac{1}{2}K+\tfrac{1}{2}iK') = \frac{k'^2}{k^2}\frac{-k+S+ik'C}{k+S-ik'C},$$

$$\mathrm{dn}^4(u+\tfrac{1}{2}K+\tfrac{1}{2}iK') = -k'^2\frac{k-S+ik'C}{k+S-ik'C}.$$

THE METHOD OF PERSPECTIVE: "WAS IT KNOWN TO THE GREEK GEOMETERS"?

By *C. Taylor, M.A.*

THE two books of Serenus of Antissa *De Sect. Cylindri* and *De Sect. Coni* respectively form a sequel to the Conics of Apollonius in Halley's edition. Serenus was also a commentator on Apollonius, and he lived before Marinus, a disciple of Proclus.* Many geometers in his day imagining that the sections of the cylinder were not identical with the elliptic sections of the cone, he sets to work to remove this misapprehension. (*De Sect. Cyl.* props. 16–18). He then shews how to construct a cylinder and a cone on coplanar bases, so as to be cut by one plane in the same ellipse or in similar ellipses (props. 19–23); and further, given a cylinder and a plane cutting it, he describes a cone having the same base and altitude, which is cut by the given plane in a section similar to that of the cylinder (prop. 25). Props. 26–30 shew how to cut a cylinder or cone in subcontrary pairs of similar ellipses. The remaining propositions *De Sect. Cylindri*, although of still greater interest and importance, are sometimes overlooked. The property of a harmonic pencil, indirectly stated, is applied *in space* to prove that all the tangents to a cone from one point have their points of contact on two generating lines (props. 33, 34), and the idea of projection by rays emitted from a luminous point is suggested and illustrated by a simple case (prop. 35). In the book *De Sect. Coni* he breaks (as he tells us) new ground, in thoroughly discussing the triangular section determined by an arbitrary plane through the vertex. Thus he makes a step towards the generalisation of Desargues, who drew his planes of section without reference to the fixed "triangle through the axis."

The writings of Serenus suggest an answer to the question (Chasles *Aperçu historique* p. 74, 1875), Was the method of perspective† known and used by the ancients? Certainly not by those who doubted whether the sections of a cylinder were also sections of a cone. But Serenus now shews

* The date 450 A.D. for Serenus may serve as a conjectural lower limit. Baldi, *Cronica de Matematici* p. 59 (Urbino 1707), boldly assigns to him the precise date 462 A.D. Suter *Gesch. der math. Wissenschaften.* I. p. 92 (Zürich 1873) prefers the date 200–100 B.C.

† For some information about perspective see Poudra's *Histoire de la Perspective ancienne et moderne* (Paris 1864).

that the property of a harmonic range may be transferred by central projection from plane to plane, and hence that any tangent to a conic section and its point of contact project into a tangent and its point of contact on any plane. The principle of perspective had thus been laid down, as the modern reader clearly sees; but if the ancients had still (as in the time of Apollonius) no complete theory of polars with respect to a conic, and if they had not learned to look upon parallel straight lines as concurrent (Chasles *Les Porismes* p. 104), the method could not have been applied by them to much effect, had it been even more distinctly formulated than by Serenus.

The preceding paragraphs had been already printed as part of the *Prolegomena* to a work which is shortly to appear, when the recently published first volume of Moritz Cantor's *Vorlesungen über Geschichte der Mathematik* (Leipzig, 1880) came under my notice. This work on the history of mathematics is in many respects more complete than its predecessors, and on turning to the pages 347–349 relating to Serenus, I found that his treatises were described somewhat more in detail, and in particular that notice was taken of the propositions 33–4 above mentioned, but in such a way as to shew that the chief point had been missed. Prop. 34 is first stated, and then it is added: "Endlich sei bemerkt dass im Satz 33 ganz gelegentlich die Grundlage zu dem geschaffen wird, was mit modernem Namen die Lehre von den Harmonikalen genannt zu werden pflegt." That this harmonic property should have appeared in Serenus is nothing remarkable, since it was known (according to Pappus) to Euclid himself; but the really noteworthy fact is that Serenus applies it *in space*, viz. to prove prop. 34, and does not introduce it "ganz gelegentlich," in an altogether aimless and casual way.

On the much disputed date of Serenus see Cantor p. 347, where it is remarked: "Sicherlich nach Christi Geburt lebte Serenus &c." It may be remarked further, by the way, that the perplexing word *ὠκυτόβοον*, as a title of one of the works of Apollonius of Perga, is in Cantor's *Vorlesungen* (pp. 297–8) replaced by *ὠκυτόκιον*, on the satisfactory authority of two Paris manuscripts; but no notice (so far as I have observed) is taken of the fact that Halley long ago, in the preface to his Apollonii *Conica* &c. (Oxon. 1710), acutely conjectured that *ὠκυτόκιον*, quick multiplier, was the true title of the work called *ὠκυτόβοον*.

ON A SPECIAL FORM OF LAPLACE'S EQUATION.

By *W. D. Niven, M.A.*

1. If we put

$$x = (c + r\cos\theta)\cos\phi,$$

$$y = (c + r\cos\theta)\sin\phi,$$

$$z = r\sin\theta,$$

then the position of any point in space may be supposed to be specified by the three coordinates θ, ϕ, r. These quantities are the parameters of three families of surfaces cutting orthogonally, viz. a cone having its vertex in the axis of z and passing through a fixed circle of radius c in the plane of x, y; a plane passing through the axis of z; and an anchor ring whose axis is the aforesaid circle. Laplace's equation referred to these coordinates becomes

$$\frac{d}{dr} r\,(c + r\cos\theta)\frac{dV}{dr}$$

$$+ \frac{1}{r}\frac{d}{d\theta}(c + r\cos\theta)\frac{dV}{d\theta}$$

$$+ \frac{r}{c + r\cos\theta}\frac{d^2V}{d\phi^2} = 0.$$

2. If we put $\rho = \log r$, and suppose that V has the same values for all longitudes, we may write the above equation thus:

$$\frac{d}{d\rho}(c + e^{\rho}\cos\theta)\frac{dV}{d\rho} + \frac{d}{d\theta}(c + e^{\rho}\cos\theta)\frac{dV}{d\theta} = 0.$$

3. We may obtain an apparently simpler form by substituting ξ and η for

$$c + e^{\rho+\theta j}, \quad c + e^{\rho-\theta j},$$

viz. we get

$$2(\xi + \eta)\frac{d^2V}{d\xi\, d\eta} + \frac{dV}{d\xi} + \frac{dV}{d\eta} = 0.$$

This equation does not yield to Lagrange's method given in Boole's *Differential Equations*, XVII. § 14, but by making another set of substitutions, viz.

$$x+y=e^{v},$$
$$x-y=t,$$

we may bring it to the form

$$\frac{d^2V}{dv^2}-e^{2v}\frac{d^2V}{dt^2}=0.$$

This admits of a solution in series of a somewhat complicated form, as in the example given by Boole, XVII. § 10.

The solution in series has been given by Riemann (*Gesammelte mathematische Werke*, Chapter XXIV.) in a form suitable to questions connected with the anchor ring.

4. My object, however, is not to enter upon the general solution, but to point out a particular case.

If we enter the term $A_n e^{m\rho}\cos n\theta$ in the equation of § 2 above, the result will be

$$A_n\left[c\,(m^2-n^2)\,e^{m\rho}\cos n\theta\right.$$
$$+\tfrac{1}{2}\{m\,(m+1)-n\,(n+1)\}\,e^{(m+1)\rho}\cos(n+1)\,\theta$$
$$\left.+\tfrac{1}{2}\{m\,(m+1)-n\,(n-1)\}\,e^{(m+1)\rho}\cos(n+1)\,\theta\right].$$

Now, by properly choosing m we can make one of these three terms vanish, and by making the signs of the coefficients A alternately positive and negative we can determine the A's, so that when $\Sigma A_n e^{m\rho}\cos n\theta$ is entered in the equation the whole series of terms will cut out.

5. I. Let $m=-n-1$, then

$$V=A\left(\frac{1}{r}-\frac{c}{r^2}\cos\theta+\&c.+(-1)^n\frac{1.3\ldots(2n-1)}{n\,!}\,\frac{c^n}{r^{n+1}}\cos n\theta+\ldots\right)$$
$$=\tfrac{1}{2}\frac{A}{r}\left\{\left(1+2\,\frac{c}{r}\,e^{\phi i}\right)^{-\frac{1}{2}}+\left(1+2\,\frac{c}{r}\,e^{-\phi i}\right)^{-\frac{1}{2}}\right\}.$$

If r' be the distance of a point P from the opposite edge of the circle in the same longitudinal plane ϕ, and if ψ denote the angle between r and r', the above expression will reduce to

$$V=A\,\frac{\cos\frac{1}{2}\psi}{\sqrt{(rr')}}.$$

The equation $V = A \cos\frac{1}{2}\psi : \sqrt{(rr')}$ considered as the equation of a system of equipotential surfaces shows that in their general character they must resemble confocal oblate ellipsoids of revolution, approximating nearer and nearer to the spherical form the further they are from the circle of reference.

The foregoing solution has been obtained for values of r greater than c, but the final form of the result shows that this restriction is unnecessary. We shall therefore assume that the solution is universally true.

In like manner we have also the solution

$$V = A \frac{\sin\frac{1}{2}\psi}{\sqrt{(rr')}},$$

and therefore by a combination of the two

$$V = A \frac{\sin\frac{1}{2}(\psi + \alpha)}{\sqrt{(rr')}},$$

The last equation solves the problem of electrical distribution on a freely charged conductor in the form of a ring, though not the ordinary anchor ring. Thus if $\alpha = \frac{1}{2}\pi$ and the linear dimensions of the section be small in comparison with c the section will resemble an ellipse whose major axis is double its minor, the latter axis being in the plane of the circle of reference. By giving different values to α we may possibly approximate to circular section.

6. II. Next let $m = n - 1$. This will lead to

$$V = A\left\{\cos\theta - \tfrac{1}{3}\frac{r}{c}\cos 2\theta + \frac{1.2}{3.5}\frac{r^2}{c^2}\cos 2\theta - \&c.\right.$$

$$\left. + (-1)^n \frac{n!\,n!}{2n+1!}\left(\frac{2r}{c}\right)^n \cos(n+1)\,\theta + \ldots\right\}.$$

This case is interesting only on account of the mathematical difficulty of summing the series

$$\sigma = 1 - \tfrac{1}{3}\alpha + \frac{1.2}{3.5}\alpha^2 - \ldots,$$

upon which summation the possibility of finding V in finite terms obviously depends.

If we denote by D the operator $\frac{d}{dx}$ and by D' the operator $\frac{d}{dy}$, we get

$$\sigma = \left\{1 + \tfrac{1}{2}\alpha\frac{D}{D'} + (\tfrac{1}{2}\alpha)^2\frac{D}{D'} + \ldots\right\}\frac{y^{\frac{1}{2}}}{x}$$

$$= \frac{1}{1 - \frac{1}{2}\alpha\frac{D}{D'}}\frac{y^{\frac{1}{2}}}{x},$$

where, after the operations are performed, we are to put $x = y = 1$.

We find

$$\sigma = \tfrac{1}{2}e^{\frac{1}{2}xyD}\int e^{-\frac{1}{2}xyD}\frac{1}{x}\frac{dy}{y^{\frac{1}{2}}}$$

$$= \tfrac{1}{2}e^{\frac{1}{2}\alpha yD}\int\frac{1}{x - \frac{1}{2}\alpha y}\frac{dy}{y^{\frac{1}{2}}}$$

$$= \tfrac{1}{2}\sqrt{\left(\frac{2}{\alpha x}\right)}e^{\frac{1}{2}\alpha yD}\log\frac{x^{\frac{1}{2}} + (\frac{1}{2}\alpha y)^{\frac{1}{2}}}{x^{\frac{1}{2}} - (\frac{1}{2}\alpha y)^{\frac{1}{2}}}$$

$$= \frac{1}{\sqrt{\{\alpha(\alpha+2)\}}}\log\frac{\sqrt{(2+\alpha)} + \sqrt{\alpha}}{\sqrt{(2+\alpha)} - \sqrt{\alpha}}.$$

By using this result we get two solutions of V from the series at the beginning of this article, viz.

$$V = \frac{2c}{\sqrt{(rr')}}\left\{\cos\tfrac{1}{2}\psi\log\frac{R}{(2c)^{\frac{1}{2}}} - \sin\tfrac{1}{2}\psi\cdot\chi\right\}$$

and

$$\frac{2c}{\sqrt{(rr')}}\left\{\sin\tfrac{1}{2}\psi\log\frac{R}{(2c)^{\frac{1}{2}}} + \cos\tfrac{1}{2}\psi\cdot\chi\right\},$$

where

$$R^2 = r + r' + 2\sqrt{(rr')}\cos\tfrac{1}{2}\psi,$$

and

$$\tan\chi = \frac{r^{\frac{1}{2}}\sin\frac{1}{2}\theta + r'^{\frac{1}{2}}\sin\frac{1}{2}\theta'}{r^{\frac{1}{2}}\cos\frac{1}{2}\theta + r'^{\frac{1}{2}}\cos\frac{1}{2}\theta'}$$

$$= \left(\frac{r - r' + 2c}{-r + r' + 2c}\right).$$

MATHEMATICAL NOTES.

An expansion for $\int_0^\infty \frac{\phi(ax)-\phi(bx)}{x^n}dx$, *n being a positive integer.*

1. THE expansion is

$$\int_b^a da \int_0^\infty \frac{\phi'(x)}{x^{n-1}}dx + \int_b^a\int_1^a da^2 \int_0^\infty \frac{\phi''(x)}{x^{n-2}}dx$$

$$+ \int_b^a\int_1^a\int_1^a da^3 \int_0^\infty \frac{\phi'''(x)}{x^{n-3}}dx + \ldots + \int_b^a\int_1^a\int_1^a \ldots da^{n-1}\int_0^\infty \frac{\phi^{n-1}(x)}{x}dx$$

$$+ \int_b^a\int_1^a\int_1^a \ldots \frac{da^n}{a}\{\phi^{n-1}(\infty)-\phi^{n-1}(0)\},$$

the form of ϕ being such as to make it intelligible.

To prove the theorem, take

$$\int_0^\infty \frac{\phi(ax)-\phi(bx)}{x}dx = \int_0^\infty\int_b^a \phi'(ax)\,da\,dx$$

$$= \{\phi(\infty)-\phi(0)\}\int_b^a \frac{da}{a},$$

transpose $\int_0^\infty \frac{\phi(bx)}{x}dx$ to the right, put 1 for b, ϕ' for ϕ, and integrate with regard to a between limits b and a, thus getting

$$\int_0^\infty \frac{\phi(ax)-\phi(bx)}{x^2}dx = \int_b^a da\int_0^\infty \frac{\phi'(x)}{x}dx$$

$$+ \int_b^a\int_1^a \frac{da^2}{a}\{\phi'(\infty)-\phi'(0)\}.$$

Again in this, transpose, put 1 for b, Q' and Q'' for Q and Q', and integrate for a as before: repeat the same process $n-1$ times, and the general result is obtained.

For the case $n=2$ the value of the integral is obtained in Bertrand's *Integral Calculus* (p. 225), but his form of result has many exceptional cases that are included in the above.

2. In the general expansion put $b=1$; then we may write the equality

$$\int_0^\infty \{\phi(ax)-\phi(x)-(a-1)x\phi'(x)-\frac{(a-1)^2x^2}{1.2}\phi''(x)-\ldots$$

$$-\frac{(a-1)^{n-1}x^{n-1}}{(n-1)!}\phi^{n-1}(x)\}\frac{dx}{x^n}=\int_b^a\int_1^a\int_1^a\ldots\frac{da^n}{a}\{\phi^{n-1}(\infty)-\phi^{n-1}(0)\}.$$

Hence if $\frac{(a-1)^nx^n}{n!}\phi^n\{x+\theta(a-1)x\}$ be the remainder after n terms of the expansion of $\phi(ax)$ in powers of $(a-1)x$, we have the equality

$$\int_0^\infty \phi^n\{x+\theta(a-1)x\}\,dx$$

$$=\frac{n!}{(a-1)^n}\int_1^a\int_1^a\int_1^a\ldots\frac{da^n}{a}\{\phi^{n-1}(\infty)-\phi^{n-1}(0)\}.$$

E. B. ELLIOTT.

Application of Lamé's coordinates to determine the distribution of electricity on a conductor in the form of an ellipsoid placed in a field of electric force.

THE axes of the ellipsoid being taken as coordinate axes, and the coordinates of the centre being meanwhile denoted by f, g, h, we may write the potential at any point inside of the conductor due to the electricity outside in the form

$$V_0+\frac{dV}{df}x+\frac{dV}{dg}y+\frac{dV}{dh}z+\tfrac{1}{2}\left(\frac{d^2V}{df^2}x^2+\frac{d^2V}{dg^2}y^2+\frac{d^2V}{dh^2}z^2\right)$$

$$+\frac{d^2V}{dg\,dh}yz+\frac{d^2V}{dh\,df}zx+\frac{d^2V}{df\,dg}xy,$$

terms of a higher order being neglected.

The potential due to the induced electricity is for inside points the same as the expression given above with its sign changed, and to solve the problem proposed to the extent of the terms given we must be able to find the potential at outside points corresponding to those terms. In finding the latter potential we may take the above terms singly or in groups as suits our purpose, always on the understanding that each single term or group of terms so taken must satisfy Laplace's equation.

It is easily seen that there is no difficulty except with the group

$$\tfrac{1}{2}\left(\frac{d^2V}{df^2}x^2+\frac{d^2V}{dg^2}y^2+\frac{d^2V}{dh^2}z^2\right).$$

To find whether this can be thrown into the form suitable to the application of Lamé's method let us examine whether it can be expressed in the form:

$$A\left(\frac{x^2}{a^2+\theta_1}+\frac{y^2}{b^2+\theta_1}+\frac{z^2}{c^2+\theta_1}-1\right)$$
$$+B\left(\frac{x^2}{a^2+\theta_2}+\frac{y^2}{b^2+\theta_2}+\frac{z^2}{c^2+\theta_2}-1\right)$$
$$+A+B.$$

It is obvious the terms as thus exhibited will be identical with the group in question, provided θ_1 and θ_2 are the roots of the equation

$$\frac{1}{a^2+\theta}+\frac{1}{b^2+\theta}+\frac{1}{c^2+\theta}=0,$$

and A, B will be determined from the equations:

$$\frac{A}{a^2+\theta_1}+\frac{B}{a^2+\theta_2}=\tfrac{1}{2}\frac{d^2V}{df^2},$$
$$\frac{A}{b^2+\theta_1}+\frac{B}{b^2+\theta_2}=\tfrac{1}{2}\frac{d^2V}{dg^2}.$$

The arrangement of the terms in the above form was suggested by §29 of Mr. Ferrers's chapter on ellipsoidal coordinates. Adopting Mr. Ferrers's notations we observe that the group of terms under discussion is equal to

$$A\frac{(\varepsilon-\theta_1)(\upsilon-\theta_1)(\upsilon'-\theta_1)}{(a^2+\theta_1)(b^2+\theta_1)(c^2+\theta_1)}$$
$$+B\frac{(\varepsilon-\theta_2)(\upsilon-\theta_2)(\upsilon'-\theta_2)}{(a^2+\theta_2)(b^2+\theta_2)(c^2+\theta_2)}$$
$$+A+B,$$

which is in Lamé's form. For brevity's sake write this

$$Af(\theta_1)+Bf(\theta_2)+A+B.$$

Putting α for Lamé's first ellipsoidal function, viz.

$$\int_{\varepsilon}^{\infty} \frac{d\varepsilon}{\sqrt{(\varepsilon+a^2)(\varepsilon+b^2)(\varepsilon+c^2)}},$$

and writing α_0 for α when $\varepsilon=0$ we can now exhibit the outside potential due to the induced electricity as minus the following terms:

$$(V_0+A+B)\,\alpha : \alpha_0$$

$$+\frac{dV}{df}x\frac{d\alpha}{da^2} : \frac{d\alpha_0}{da^2} + \text{two similar terms}$$

$$+Af(\theta_1)\int_{\varepsilon}^{\infty} \frac{-\frac{d\alpha}{d\varepsilon}}{(\varepsilon-\theta_1)^2}d\varepsilon : \int_{0}^{\infty} \frac{-\frac{d\alpha}{d\varepsilon}}{(\varepsilon-\theta_1)^2}d\varepsilon + \text{one similar term}$$

$$+\frac{d^2V}{dfdg}xy\frac{d^2\alpha}{da^2db^2} : \frac{d^2\alpha_0}{da^2db^2} + \text{two similar terms.}$$

The problem of finding the induced magnetization of a solid ellipsoid of soft iron placed in a field of magnetic force can obviously be carried to the same degree of approximation.

W. D. NIVEN.

On the potential at any point of space due to a solid sphere whose density varies inversely as the fifth power of the distance from a fixed point.

I BEGIN first with a solid homogeneous sphere whose centre is A, radius $=a$, and mass $=m$. The well-known values of the potential at any point P due to this sphere, may be exhibited in the form

$$\Sigma\frac{\rho dv}{QP} = \frac{m}{AP} \text{ or } \frac{m}{2a^3}(3a^2-AP^2),$$

according as P is outside or inside the sphere, dv being an element of the volume at Q.

Now let us take a point C (fig. 5) at distance c from the centre, and invert the sphere with regard to the point C into itself, so that the tangent $CT=\sqrt{(c^2-a^2)}$, $=R$ suppose, is the radius of inversion. The point A obviously inverts into the point a, where TaC is a right angle.

It will be convenient to recall the rule of inversion. If r be the distance from the point of inversion, the potential for the inverted figure may be derived from the corresponding potential in the original figure, by merely changing in the latter $r : R$ into $R : r$, and multiplying the result by $R : r$.

Now if we merely make the change $r : R$ into $R : r$ in the expression for any line AP, viz. $\sqrt{(c^2 - 2cr\mu + r^2)}$ we get $ap.c : r$ where p is the inverse point of P, as a is of A. In like manner QP becomes $qp.CQ : r$,

Entering these changes in the expressions with which we set out for a homogeneous sphere, and recollecting that $dv : dv' = CQ^3 : Cq^3 = R^5 : Cq^5$, where dv' is the inverted volume corresponding to dv, we find

$$\Sigma \frac{\rho R^5 : Cq^5}{qp} dv' = \frac{m.R : c}{ap},$$

or

$$\frac{R}{r} \frac{m}{2a^3} \left(3a^2 - c^2 + 2c \frac{R^2}{r} \mu - \frac{R^4}{r^2}\right),$$

according as p is outside or inside of the new mass.

These equations shew that we have obtained at all points of space the potential due to a sphere whose density varies as the inverse fifth power of the distance from C. If we denote the mass of the new sphere by M, then recollecting that $m = \frac{4}{3}\pi a^3 \rho$, we see that its density at any point q is

$$\frac{3}{4\pi} \frac{c\,(c^2 - a^2)^2}{a^3 Cq^5} M.$$

These are well-known results.

W. D. NIVEN.

Proof of Euclid I. 25.

I discovered nearly five years ago the following direct proof of Euc. I. 25:

Let ABC, DEF be the two triangles such that $AB = DE$, $AC = DF$, but $BC > EF$. Then shall $\angle BAC > \angle EDF$.

Place the triangle DEF in such a manner, that E may coincide with B, EF may fall on BC, and BD may be conterminous with AB. Join AD. Let DF and AC meet each other at G.

Now $BA = ED = BD$ (Hyp.);

therefore $\angle BAD = \angle BDA$ (Euc. I. 5).

Again (fig. 6) $DG > DF > AC > AG$.

therefore $\angle DAG > \angle ADG$.

therefore $\angle BAC > \angle BDF$, or $\angle BAC > \angle EDF$.

Again (fig. 7), it may similarly be shewn that

$$\angle ADG > \angle DAG;$$

therefore $\angle BAC > \angle BDF$,

or $\angle BAC > EDF$. Q.E.D.

N.B. This method of proof is an instance of how powerful an instrument the method of superposition is. It has already been exemplified in the new proof of Euc. I. 8, which dispenses with the clumsy proposition Euc. I. 7. As has already been noticed, the proof of Euc. I. 26, can be much simplified in this way. (Todhunter's *Euclid*, p. 382, Ex. 448).

ASUTOSH MUKHOPADHYAY.

Bhowanspore, near Calcutta,
Oct. 6th, 1880.

TRANSACTIONS OF SOCIETIES.

London Mathematical Society.

Thursday, Nov, 4, 1880, C. W. Merrifield, F.R.S., *President*, in the Chair. The Treasurer's and Secretaries' Reports were read and adopted. The meeting then proceeded to ballot, when the following gentlemen were elected on the Council for the present Session: *President*—S. Roberts, F.R.S.; *Vice-Presidents*—Dr. Hirst, F.R.S., and J. W. L. Glaisher, F.R.S.; *Treasurer*—C. W. Merrifield, F.R.S.; *Hon. Secretaries*—M. Jenkins, M.A., and R. Tucker, M.A.; *Other members*—Prof. Cayley, F.R.S.; H. Hart, M.A.; Dr. Henrici, F.R.S.; Dr. Hopkinson, F.R.S.; A. B. Kempe, B.A.; R. F. Scott, M.A.; Prof. H. J. S. Smith, F.R.S.; H. W. Lloyd Tanner, M.A.; H. M. Taylor, M.A.; and J. J. Walker, M.A. Mr. Roberts having taken the Chair, Mr. Merrifield read his valedictory address, "Considerations respecting the translation of Series of Observations into Continuous Formulæ." On the motion of Prof. Cayley, the address was ordered to be printed in the *Proceedings*. Mr. H. M. Jeffery, F.R.S., then read a paper on "Bicircular Quartics, with a Triple and Double Focus and Three Single Foci, all of them collinear." Mr. Tucker read parts of a paper by the Mr. C. Taylor, M.A., "Further remarks on the Geometrical Method of Reversion."

Thursday, Dec. 9. 1880, Mr. S. Roberts, F.R.S., *President*, in the Chair. Messrs. W. R. Roberts, M.A., Ex-Scholar, and R. A. Roberts, B.A., Scholar, of Trinity College, Dublin, were elected Members. The following communications were made: "Note sur la détermination des Déterminants," Prof. Teixeira (Coimbra, Portugal); "Solution of the Binomial Equation $x^p - 1 = 0$, quinquisection," Prof. Cayley, F.R.S.; "A general theorem in Kinematics," Prof. Minchin, M.A.; "On the Solution of the Inverse Logical Problem," Mr. W. B. Grove, B.A.; "Motion of a Viscous Fluid," Mr. T. Craig; "On the Electrical Capacity of a Conductor bounded by two Spherical Surfaces cutting at any Angle," Mr. W. D. Niven, M.A.

R. TUCKER, M.A., *Hon. Sec.*

[illegible] THE [illegible] EQUATION FOR THE [illegible] INTEGRAL.

By [illegible] *Glaisher*.

[illegible] The addition equation for the third elliptic integral is

$$\Pi(u+v,\alpha)-\Pi(u-v,\alpha)-\Pi(a-u,\alpha)-\Pi(a-v,\alpha)=\tfrac{1}{2}\log\Omega,$$

where
$$\Pi=\frac{1-\frac{1}{2}\operatorname{sn}^2 a\operatorname{sn}^2 u\operatorname{sn}^2 v\operatorname{sn}^2(u+v-a)}{1-\frac{1}{2}\operatorname{sn}^2 a\operatorname{sn}^2(u-a)\operatorname{sn}^2(v-a)\operatorname{sn}^2(u+v-a)} \quad \text{.....(1).}$$

This value of Π is equivalent to that given by Legendre [illegible] with Π.

In the Fundamenta [illegible], Jacobi arrived at two other expressions for Π, viz.

$$\frac{1-\frac{1}{2}\operatorname{sn}^2 a\{\operatorname{sn}^2(u+v)+\operatorname{sn}^2(u-v)-\operatorname{sn}^2 u-\operatorname{sn}^2 v+\operatorname{sn}^2\tfrac{1}{2}(u+v+a)+\operatorname{sn}^2\tfrac{1}{2}(u+v-a)\}}{1-\frac{1}{2}\operatorname{sn}^2 a\{\operatorname{sn}^2(u+v)+\operatorname{sn}^2(u-v)-\operatorname{sn}^2 u-\operatorname{sn}^2 v-\operatorname{sn}^2\tfrac{1}{2}(u+v+a)-\operatorname{sn}^2\tfrac{1}{2}(u+v-a)\}}$$

.....(2),

$$\left.\begin{array}{l}1-\frac{1}{2}\operatorname{sn}^2 a\operatorname{sn}^2 u\operatorname{sn}^2 v\operatorname{sn}^2(u+v-a)\\ 1-\frac{1}{2}\operatorname{sn}^2 a\operatorname{sn}^2(u-a)\operatorname{sn}^2(v-a)\operatorname{sn}^2(u+v-a)\end{array}\right\}$$

.....(3).

Professor [illegible] Smith has also given an expression for Π which will be referred to in § 4.

§ 2. I have obtained a system of 96 expressions for Ω in all of which, as in those given above, the numerator and denominator differ only by the change of sign of a. This system, which includes [illegible] is given below.

For the sake of brevity s_1, s_2, s_3, s_4 are written in place of $\operatorname{sn} u, \operatorname{cn} u, \operatorname{dn} u$ and the suffixes 1, 2, ...6 refer to the arguments $u, v, \ldots v+a$ according to the scheme.

1	u
2	v
3	a
4	$u+v-a$
5	$u+a$
6	$v+a$

so that s_3, c_4, d_6, &c. denote $\mathrm{sn}\,a$, $\mathrm{cn}\,(u+v-a)$, $\mathrm{dn}\,(v+a)$, &c. respectively.

An accent attached to a letter denotes that the sign of a in the argument is changed, so that when the letters are accented the arguments corresponding to 4, 5, 6 (the only suffixes attached to accented letters), are

$$u+v+a,\quad u-a,\quad v-a,$$

respectively. Thus s'_4, c'_5, &c. denote $\mathrm{sn}\,(u+v+a)$, $\mathrm{cn}\,(u-a)$, &c.

The expressions for Ω are arranged in 48 pairs numbered (i), (ii), ...(xlviii); the connexion between the members of each pair will be explained in § 4.

No accented letters occur in the numerators; in the denominators the letters having suffixes 4, 5, 6 are always accented. This, with the change of sign of s_3 (and, when the expression is negative, of the whole denominator also), constitutes the only difference between the denominator and numerator in each expression.

Expressions for Ω.

(i) $$\frac{1-k^2s_1s_2s_3s_4}{1+k^2s_1s_2s_3s_4'},\qquad \frac{d_1d_2d_3d_4-k^2c_1c_2c_3c_4}{d_1d_2d_3d_4'-k^2c_1c_2c_3c_4'},$$

(ii) $$\frac{(k'^2-d_1d_2d_3d_4)\,c_5c_6}{(k'^2-d_1d_2d_3d_4')\,c_5'c'},\qquad \frac{(c_1c_2c_3c_4-k^2s_1s_2s_3s_4)\,c_5c_6}{(c_1c_2c_3c_4'+k'^2s_1s_2s_3s_4')\,c_5'c_6'},$$

(iii) $$\frac{(k'^2+k^2c_1c_2c_3c_4)\,d_5d_6}{(k'^2+k^2c_1c_2c_3c_4')\,d_5'd_6'},\qquad \frac{(d_1d_2d_3d_4+k^2k'^2s_1s_2s_3s_4)\,d_5d_6}{(d_1d_2d_3d_4'-k^2k'^2s_1s_2s_3s_4')\,d_5'd_6'},$$

(iv) $$\frac{(s_1s_2-s_3s_4)\,s_5s_6}{(s_1s_2+s_3s_4')\,s_5's_6'},\qquad \frac{(c_1c_2d_3d_4-c_3c_4d_1d_2)\,s_5s_6}{(c_1c_2d_3d_4'-c_3c_4'd_1d_2)\,s_5's_6'},$$

(v) $$-\frac{(s_1s_3-s_2s_4)\,s_6}{(s_1s_3+s_2s_4')\,s_6'},\qquad \frac{(c_1c_3d_2d_4-c_2c_4d_1d_3)\,s_6}{(c_1c_3d_2d_4'-c_2c_4'd_1d_3)\,s_6'},$$

(vi) $$\frac{(s_1s_4-s_2s_3)\,s_5}{(s_1s_4'+s_2s_3)\,s_5'},\qquad \frac{(c_1c_4d_2d_3-c_2c_3d_1d_4)\,s_5}{(c_1c_4'd_2d_3-c_2c_3d_1d_4')\,s_5'},$$

(vii) $$\frac{(c_1c_2-c_3c_4)\,s_5s_6}{(c_1c_2-c_3c_4')\,s_5's_6'},\qquad \frac{(s_1s_2d_3d_4-s_3s_4d_1d_2)\,s_5s_6}{(s_1s_2d_3d_4'+s_3s_4'd_1d_2)\,s_5's_6'},$$

(viii) $$\frac{(c_1c_3 - c_2c_4)\, s_6 d_5}{(c_1c_3 - c_2c_4')\, s_6' d_5'}, \qquad -\frac{(s_1s_3d_2d_4 - s_2s_4 d_1d_3)\, s_6 d_5}{(s_1s_3d_2d_4 + s_2s_4' d_1d_3)\, s_6' d_5'},$$

(ix) $$\frac{(c_1c_4 - c_2c_3)\, s_5 d_6}{(c_1c_4' - c_2c_3)\, s_5' d_6'}, \qquad \frac{(s_1s_4 d_2d_3 - s_2s_3 d_1d_4)\, s_5 d_6}{(s_1s_4' d_2d_3 + s_2s_3 d_1d_4')\, s_5' d_6'},$$

(x) $$\frac{(d_1d_2 - d_3d_4)\, s_5 s_6}{(d_1d_2 - d_3d_4')\, s_5' s_6'}, \qquad \frac{(s_1s_2c_3c_4 - s_3s_4 c_1c_2)\, s_5 s_6}{(s_1s_2c_3c_4' + s_3s_4' c_1c_2)\, s_5' s_6'},$$

(xi) $$\frac{(d_1d_3 - d_2d_4)\, s_6 c_5}{(d_1d_3 - d_2d_4')\, s_6' c_5'}, \qquad -\frac{(s_1s_3c_2c_4 - s_2s_4 c_1c_3)\, s_6 c_5}{(s_1s_3c_2c_4' + s_2s_4' c_1c_3)\, s_6' c_5'},$$

(xii) $$\frac{(d_1d_4 - d_2d_3)\, s_5 c_6}{(d_1d_4' - d_2d_3')\, s_5' c_6'}, \qquad \frac{(s_1s_4 c_2c_3 - s_2s_3 c_1c_4)\, s_5 c_6}{(s_1s_4' c_2c_3 + s_2s_3 c_1c_4')\, s_5' c_5'},$$

(xiii) $$\frac{(k'^2 s_1s_2 - c_1c_2d_3d_4)\, d_5 d_6}{(k'^2 s_1s_2 - c_1c_2d_3d_4')\, d_5' d_6'}, \qquad -\frac{(k'^2 s_3s_4 - c_3c_4 d_1d_2)\, d_5 d_6}{(k'^2 s_3s_4' + c_3c_4' d_1d_2)\, d_5' d_6'},$$

(xiv) $$\frac{(k'^2 s_1s_2 + c_3c_4 d_1d_2)\, c_5 c_6}{(k'^2 s_1s_2 + c_3c_4' d_1d_2)\, c_5' c_6'}, \qquad -\frac{(k'^2 s_3s_4 + c_1c_2d_3d_4)\, c_5 c_6}{(k'^2 s_3s_4' - c_1c_2d_3d_4')\, c_5' c_6'},$$

(xv) $$-\frac{(k'^2 s_1s_3 + c_1c_3d_2d_4)\, c_5 d_6}{(k'^2 s_1s_3 - c_1c_3d_2d_4')\, c_5' d_6'}, \qquad \frac{(k'^2 s_2s_4 + c_3c_4 d_1d_3)\, c_5 d_6}{(k'^2 s_2s_4' + c_2c_4' d_1d_3)\, c_5' d_6'},$$

(xvi) $$-\frac{(k'^2 s_1s_3 - c_2c_4 d_1d_3)\, c_6 d_5}{(k'^2 s_1s_3 + c_2c_4' d_1d_3)\, c_6' d_5'}, \qquad \frac{(k'^2 s_2s_4 - c_1c_3d_2d_4)\, c_6 d_5}{(k'^2 s_2s_4' - c_1c_3d_2d_4')\, c_6' d_5'},$$

(xvii) $$\frac{(k'^2 s_1s_4 + c_1c_4 d_2d_3)\, c_6 d_5}{(k'^2 s_1s_4' + c_1c_4' d_2d_3)\, c_6' d_5'}, \qquad -\frac{(k'^2 s_2s_3 + c_2c_3 d_1d_4')\, c_6 d_5}{(k'^2 s_2s_3 - c_2c_3 d_1d_4')\, c_6' d_5'},$$

(xviii) $$\frac{(k'^2 s_1s_4 - c_2c_3 d_1d_4)\, c_5 d_6}{(k'^2 s_1s_4' - c_2c_3 d_1d_4')\, c_5' d_6'}, \qquad -\frac{(k'^2 s_2s_3 - c_1c_4 d_2d_3)\, c_5 d_6}{(k'^2 s_2s_3 + c_1c_4' d_2d_3)\, c_5' d_6'},$$

(xix) $$\frac{c_1c_2 - s_1s_2d_3d_4}{c_1c_2 - s_1s_2d_3d_4'}, \qquad \frac{c_3c_4 - s_3s_4 d_1d_2}{c_3c_4 + s_3s_4' d_1d_2},$$

(xx) $$\frac{(c_1c_2 + s_3s_4 d_1d_2)\, c_5 c_6}{(c_1c_2 - s_3s_4' d_1d_2)\, c_5' c_6'}, \qquad \frac{(c_3c_4 + s_1s_2d_3d_4)\, c_5 c_6}{(c_3c' + s_1s_2d_3d_4')\, c_5' c_6'},$$

(xxi) $$\frac{(c_1c_2+s_1s_2d_2d_4)\,c_5}{(c_1c_2-s_1s_2d_2d_4')\,c_5'}, \qquad \frac{(c_2c_4+s_2s_4d_1d_2)\,c_5}{(c_2c_4'+s_2s_4'd_1d_2)\,c_5'},$$

(xxii) $$\frac{(c_1c_3-s_2s_4d_1d_3)\,c_6'}{(c_1c_3-s_2s_4'd_1d_3)\,c_6'}, \qquad \frac{(c_2c_4-s_1s_3d_2d_4)\,c_6}{(c_2c_4'+s_1s_3d_2d_4')\,c_6'},$$

(xxiii) $$\frac{(c_1c_4+s_1s_4d_2d_3)\,c_6}{(c_1c_4'+s_1s_4'd_2d_3)\,c_6'}, \qquad \frac{(c_2c_3+s_2s_3d_1d_4)\,c_6}{(c_2c_3-s_2s_3d_1d_4')\,c_6'},$$

(xxiv) $$\frac{(c_1c_4-s_2s_3d_1d_4)\,c_5}{(c_1c_4'+s_2s_3d_1d_4')\,c_5'}, \qquad \frac{(c_2c_3-s_1s_4d_2d_3)\,c_5}{(c_2c_3-s_1s_4'd_2d_3)\,c_5'},$$

(xxv) $$\frac{d_1d_2-k^2s_1s_2c_3c_4}{d_1d_2-k^2s_1s_2c_3c_4'}, \qquad \frac{d_3d_4-k^2s_3s_4c_1c_2}{d_3d_4+k^2s_3s_4'c_1c_2},$$

(xxvi) $$\frac{(d_1d_2+k^2s_3s_4c_1c_2)\,d_5d_6}{(d_1d_2-k^2s_3s_4'c_1c_2)\,d_5'd_6'}, \qquad \frac{(d_3d_4+k^2s_1s_2c_3c_4)\,d_5d_6}{(d_3d_4'+k^2s_1s_2c_3c_4')\,d_5'd_6'},$$

(xxvii) $$\frac{(d_1d_3+k^2s_1s_3c_2c_4)\,d_6}{(d_1d_3-k^2s_1s_3c_2c_4')\,d_5'}, \qquad \frac{(d_2d_4+k^2s_2s_4c_1c_3)\,d_5}{(d_2d_4'+k^2s_2s_4'c_1c_3)\,d_5'},$$

(xxviii) $$\frac{(d_1d_3-k^2s_2s_4c_1c_3)\,d_6}{(d_1d_3-k^2s_2s_4'c_1c_3)\,d_6'}, \qquad \frac{(d_2d_4-k^2s_1s_3c_2c_4)\,d_6}{(d_2d_4'+k^2s_1s_3c_2c_4')\,d_6'},$$

(xxix) $$\frac{(d_1d_4+k^2s_1s_4c_2c_3)\,d_6}{(d_1d_4'+k^2s_1s_4'c_2c_3)\,d_6'}, \qquad \frac{(d_2d_3+k^2s_2s_3c_1c_4)\,d_6}{(d_2d_3-k^2s_2s_3c_1c_4')\,d_6'},$$

(xxx) $$\frac{(d_1d_4-k^2s_3s_3c_1c_4)\,d_5}{(d_1d_4'+k^2s_2s_3c_1c_4')\,d_5'}, \qquad \frac{(d_2d_3-k^2s_1s_4c_2c_3)\,d_5}{(d_2d_3-k^2s_1s_4'c_2c_3)\,d_5'},$$

(xxxi) $$\frac{(c_3d_4s_1+c_4d_3s_2)\,c_6d_5}{(c_3d_4's_1+c_4'd_3s_2)\,c_6'd_5'}, \qquad -\frac{(c_1d_3s_2+c_2d_1s_4)\,c_6d_5}{(c_1d_3s_2-c_2d_1s_4')\,c_6'd_5'},$$

(xxxii) $$\frac{(c_4d_3s_1+c_3d_4s_2)\,c_5d_6}{(c'd_3s_1+c_3d_4's_2)\,c_5'd_6'}, \qquad -\frac{(c_2d_1s_3+c_1d_2s_4)\,c_5d_6}{(c_2d_1s_3-c_1d_2s_4')\,c_5'd_6'},$$

(xxxiii) $$\frac{(c_2d_4s_1-c_4d_2s_3)\,s_5c_6}{(c_2d_4's_1+c_4'd_2s_3)\,s_5'c_6'}, \qquad \frac{(c_1d_2s_2-c_2d_1s_4)\,s_5c_6}{(c_1d_2s_2-c_2d_1s_4')\,s_5'c_6'},$$

(xxxiv) $$\frac{(c_4d_2s_1-c_2d_4s_3)\,s_5d_6}{(c_4'd_2s_1+c_2d_4's_3)\,s_5'd_6'}, \qquad \frac{(c_2d_1s_2-c_1d_2s_4)\,s_5d_6}{(c_2d_1s_2-c_1d_2s_4')\,s_5'd_6'},$$

(xxxv) $$\frac{(c_2d_3s_1 - c_3d_2s_4)s_6c_5}{(c_2d_3s_1 - c_3d_2s_4')s_6'c_5'}, \qquad \frac{(c_1d_4s_2 - c_4d_1s_3)s_6c_5}{(c_1d_4's_2 + c_4'd_1s_3)s_6'c_5'},$$

(xxxvi) $$\frac{(c_3d_2s_1 - c_2d_3s_4)s_6d_5}{(c_3d_2s_1 - c_2d_3s_4')s_6'd_5'}, \qquad \frac{(c_4d_1s_2 - c_1d_4s_3)s_6d_5}{(c_4d_1s_2 + c_1d_4's_3)s_6'd_5'},$$

(xxxvii) $$-\frac{(s_3d_4c_1 + s_4d_3c_2)c_6}{(s_3d_4'c_1 - s_4'd_3c_2)c_6'}, \qquad \frac{(s_1d_2c_3 + s_2d_1c_4)c_6}{(s_1d_2c_3 + s_2d_1c_4')c_6'},$$

(xxxviii) $$\frac{(s_4d_3c_1 + s_3d_4c_2)c_5}{(s_4'd_3c_1 - s_3d_4'c_2)c_5'}, \qquad \frac{(s_2d_1c_3 + s_1d_2c_4)c_5}{(s_2d_1c_3 + s_1d_2c_4')c_5'},$$

(xxxix) $$\frac{(s_2d_4c_1 - s_4d_2c_3)s_5c_6}{(s_2d_4'c_1 - s_4'd_2c_3)s_5'c_6'}, \qquad \frac{(s_1d_3c_2 - s_3d_1c_4)s_5c_6}{(s_1d_3c_2 + s_3d_1c_4')s_5'c_6'},$$

(xl) $$\frac{(s_4d_2c_1 - s_2d_4c_3)s_5}{(s_4'd_2c_1 - s_2d_4'c_3)s_5'}, \qquad -\frac{(s_3d_1c_2 - s_1d_3c_4)s_5}{(s_3d_1c_2 + s_1d_3c_4')s_5'},$$

(xli) $$\frac{(s_2d_3c_1 - s_3d_2c_4)s_6c_5}{(s_2d_3c_1 + s_3d_2c_4')s_6'c_5'}, \qquad \frac{(s_1d_4c_2 - s_4d_1c_3)s_6c_5}{(s_1d_4'c_2 - s_4'd_1c_3)s_6'c_5'},$$

(xlii) $$-\frac{(s_3d_2c_1 - s_2d_3c_4)s_6}{(s_3d_2c_1 + s_2d_3c_4')s_6'}, \qquad \frac{(s_4d_1c_2 - s_1d_4c_3)s_6}{(s_4'd_1c_2 - s_1d_4'c_3)s_6},$$

(xliii) $$-\frac{(s_3c_4d_1 + s_4c_3d_2)d_6}{(s_3c_4'd_1 - s_4'c_3d_2)d_6'}, \qquad \frac{(s_1c_2d_3 + s_2c_1d_4)d_6}{(s_1c_2d_3 + s_2c_1d_4')d_6'},$$

(xliv) $$\frac{(s_4c_3d_1 + s_3c_4d_2)d_5}{(s_4'c_3d_1 - s_3c_4'd_2)d_5'}, \qquad \frac{(s_2c_1d_3 + s_1c_2d_4)d_5}{(s_2c_1d_3 + s_1c_2d_4')d_5'},$$

(xlv) $$\frac{(s_2c_4d_1 - s_4c_2d_3)s_5d_6}{(s_2c_4'd_1 - s_4'c_2d_3)s_5'd_6'}, \qquad \frac{(s_1c_3d_2 - s_3c_1d_4)s_5d_6}{(s_1c_3d_2 + s_3c_1d_4')s_5'd_6'},$$

(xlvi) $$\frac{(s_4c_2d_1 - s_2c_4d_3)s_5}{(s_4'c_2d_1 - s_2c_4'd_3)s_5'}, \qquad -\frac{(s_3c_1d_2 - s_1c_3d_4)s_5}{(s_3c_1d_2 + s_1c_3d_4')s_5'},$$

(xlvii) $$\frac{(s_2c_3d_1 - s_3c_2d_4)s_6d_5}{(s_2c_3d_1 + s_3c_2d_4')s_6'd_5'}, \qquad \frac{(s_1c_4d_2 - s_4c_1d_3)s_6d_5}{(s_1c_4'd_2 - s_4'c_1d_3)s_6'd_5'},$$

(xlviii) $$-\frac{(s_3c_2d_1 - s_2c_3d_4)s_6}{(s_3c_2d_1 + s_2c_3d_4')s_6'}, \qquad \frac{(s_4c_1d_2 - s_1c_4d_3)s_6}{(s_4'c_1d_2 - s_1c_4'd_3)s_6'},$$

§ 3. Of these expressions only six, viz. (i), (xix), (xxv) do not involve either of the suffixes 5, 6. Omitting (i), the first form of which is (1) of § 1, the other four, written at length, are

$$\frac{\operatorname{cn}u\operatorname{cn}v-\operatorname{sn}u\operatorname{sn}v\operatorname{dn}a\operatorname{dn}(u+v-a)}{\operatorname{cn}u\operatorname{cn}v-\operatorname{sn}u\operatorname{sn}v\operatorname{dn}a\operatorname{dn}(u+v+a)},$$

$$\frac{\operatorname{cn}a\operatorname{cn}(u+v-a)-\operatorname{sn}a\operatorname{sn}(u+v-a)\operatorname{dn}u\operatorname{dn}v}{\operatorname{cn}a\operatorname{cn}(u+v+a)+\operatorname{sn}a\operatorname{sn}(u+v+a)\operatorname{dn}u\operatorname{dn}v},$$

$$\frac{\operatorname{dn}u\operatorname{dn}v-k^2\operatorname{sn}u\operatorname{sn}v\operatorname{cn}a\operatorname{cn}(u+v-a)}{\operatorname{dn}u\operatorname{dn}v-k^2\operatorname{sn}u\operatorname{sn}v\operatorname{cn}a\operatorname{cn}(u+v+a)},$$

$$\frac{\operatorname{dn}a\operatorname{dn}(u+v-a)-k^2\operatorname{sn}a\operatorname{sn}(u+v-a)\operatorname{cn}u\operatorname{cn}v}{\operatorname{dn}a\operatorname{dn}(u+v+a)+k^2\operatorname{sn}a\operatorname{sn}(u+v+a)\operatorname{cn}u\operatorname{cn}v},$$

which may be also put in the forms

$$\frac{1-\operatorname{tn}u\operatorname{tn}v\operatorname{dn}a\operatorname{dn}(u+v-a)}{1-\operatorname{tn}u\operatorname{tn}v\operatorname{dn}a\operatorname{dn}(u+v+a)},$$

$$\frac{1-\operatorname{tn}a\operatorname{tn}(u+v-a)\operatorname{dn}u\operatorname{dn}v}{1+\operatorname{tn}a\operatorname{tn}(u+v+a)\operatorname{dn}u\operatorname{dn}v},$$

$$\frac{k'^2-k^2\operatorname{cnc}u\operatorname{cnc}v\operatorname{cn}a\operatorname{cn}(u+v-a)}{k'^2-k^2\operatorname{cnc}u\operatorname{cnc}v\operatorname{cn}a\operatorname{cn}(u+v+a)},$$

$$\frac{k'^2-k^2\operatorname{cn}u\operatorname{cn}v\operatorname{cnc}a\operatorname{cnc}(u+v-a)}{k'^2+k^2\operatorname{cn}u\operatorname{cn}v\operatorname{cnc}a\operatorname{cnc}(u+v+a)},$$

where $\operatorname{cnc}x$ denotes $\operatorname{cn}(K-x)$.

§ 4. The pairs of values which appear in the same line are transformable into one another by means of the 16 equations,

$$\begin{aligned}
k'^2-k^2k'^2s_1s_2s_3s_4+k^2c_1c_2c_3c_4-d_1d_2d_3d_4&=0,\\
k'^2(s_1s_2-s_3s_4)&=\ c_1c_2d_3d_4-c_3c_4d_1d_2,\\
k'^2(s_1s_3-s_2s_4)&=-(c_1c_3d_2d_4-c_2c_4d_1d_3),\\
k'^2(s_1s_4-s_2s_3)&=-(c_1c_4d_2d_3-c_2c_3d_1d_4),\\
c_1c_2-c_3c_4&=\ s_1s_2d_3d_4-s_3s_4d_1d_2,\\
c_1c_3-c_2c_4&=-(s_1s_3d_2d_4-s_2s_4d_1d_3),\\
c_1c_4-c_2c_3&=-(s_1s_4d_2d_3-s_2s_3d_1d_4),\\
d_1d_2-d_3d_4&=\ k^2(s_1s_2c_3c_4-s_3s_4c_1c_2),\\
d_1d_3-d_2d_4&=-k^2(s_1s_3c_2c_4-s_2s_4c_1c_3),\\
d_1d_4-d_2d_3&=-k^2(s_1s_4c_2c_3-s_2s_3c_1c_4),
\end{aligned}$$

$$s_1c_2d_3+s_2c_1d_4=s_3c_4d_1+s_4c_3d_2,$$
$$s_1c_3d_4+s_2c_1d_3=s_3c_4d_2+s_4c_3d_1,$$
$$s_1c_3d_2+s_2c_4d_1=s_3c_1d_4+s_4c_2d_3,$$
$$s_1c_4d_2+s_2c_3d_1=s_3c_2d_4+s_4c_1d_3,$$
$$s_1c_3d_4+s_2c_4d_3=s_3c_1d_2+s_4c_2d_1,$$
$$s_1c_4d_3+s_2c_3d_4=s_3c_2d_1+s_4c_1d_2,$$

and of the system derived from them by changing the sign of a.

These equations are merely variations of the five formulæ,

$$(a)\quad k'^2-k^2k'^2S_1S_2S_3S_4+k^2C_1C_2C_3C_4-D_1D_2D_3D_4=0,$$
$$(b)\quad k'^2(S_1S_2-S_3S_4)-(C_1C_2D_3D_4-C_3C_4D_1D_2)=0,$$
$$(c)\quad C_1C_2-C_3C_4-(S_1S_2D_3D_4-S_3S_4D_1D_2)=0,$$
$$(d)\quad D_1D_2-D_3D_4-k^2(S_1S_2C_3C_4-S_3S_4C_1C_2)=0,$$
$$(e)\quad S_1C_2D_4+S_2C_1D_3+S_3C_4D_2+S_4C_3D_1=0,$$

where S, C, D denote sn, cn, dn and the suffixes refer to any four arguments whose sum is zero.*

The arguments

$$u,\ v,\ -a,\ -(u+v-a)$$

are such that their sum is zero, and by substituting them in (a), ...(e) we obtain the system given above.

The formula (a) is due to Gudermann,† and the other four to Professor H. J. S. Smith.‡

It will be noticed that the groups of formulæ (vii)...(ix) and (x),..(xii), (xxxvii)...(xlii) and (xliii)...(xlviii) differ from one another only by the interchange of the letters c and d. This is also the case with pairs of formulæ from (xxxi) to (xxxvi). The group (xxv)...(xxx) differs from (xix)...(xxiv) by the interchange of c and d, and the presence of the factor k^2.

§5. The 96 expressions in §2 were deduced from four systems of expressions (each 24 in number) which were

* The small letters s, c, d are used for sn, cn, dn whenever the suffixes refer to the arguments contained in the schemes in §§1 and 6; the capitals are here used to denote that the arguments are different from these.

† *Crelle's Journal*, t. xviii. p. 164; see also *Proceedings of the London Mathematical Society*, t. x. pp. 43—48.

‡ *Proceedings of the London Mathematical Society*, t. x. pp. 91—100, 231—238.

respectively equal to L, M, N, R where

$$L=\frac{(1-k^2z^4)(1-k^2x^2y^2)}{(1-k^2x^2z^2)(1-k^2y^2z^2)},$$

$$M=\frac{(1-2z^2+k^2z^4)(1-k^2x^2y^2)}{(1-k^2x^2z^2)(1-k^2y^2z^2)},$$

$$N=\frac{(1-2k^2z^2+k^2z^4)(1-k^2x^2y^2)}{(1-k^2x^2z^2)(1-k^2y^2z^2)},$$

$$R=\frac{2z\sqrt{\{(1-z^2)(1-k^2z^2)\}}(1-k^2x^2y^2)}{(1-k^2x^2z^2)(1-k^2y^2z^2)},$$

and

$$x=\operatorname{sn}\{\tfrac{1}{2}(u+v)-a\},$$
$$y=\operatorname{sn}\tfrac{1}{2}(u-v),$$
$$z=\operatorname{sn}\tfrac{1}{2}(u+v),$$

so that

$$M=L\operatorname{cn}(u+v),\quad N=L\operatorname{dn}(u+v),\quad R=L\operatorname{sn}(u+v).$$

If L', M', ... denote the values of L, M, ... when the sign of a is changed, we have

$$\frac{L}{L'}=\frac{M}{M'}=\frac{N}{N'}=\Omega.$$

The expressions in § 2 were obtained by dividing each of the 96 values of L, M, N or R by the same expression with the sign of a changed, so that in all cases the numerator and denominator are similar in form. But if this restriction be removed and any one of the 24 expressions for, say, L be divided by any one of the 24 expressions for L', the total number of expressions so obtained for Ω is $4\times(24)^2=2304$. The 2208 unsymmetrical expressions present but little interest; they are formed from the numerators and denominators of the expressions in § 3, with the occasional addition of the factors k^2, k'^2, $\frac{k^2}{k'^2}$ or their reciprocals. For example

$$\Omega=\frac{(s_1s_4-s_2s_3)\,s_6}{(s_1s_3+s_2s_4')\,s'_5}=-\frac{(s_1s_3-s_2s_4)\,s_5}{(s_1s_4'+s_2s_3)\,s'_6}$$

$$=k^2\frac{(c_1c_3-c_2c_4)\,s_6c_5}{(d_1d_3-d_2d'_4)\,s'_6d'_5}=\frac{1}{k^2}\frac{(d_1d_3-d_2d_4)\,s_6d_5}{(c_1c_3-c_2c'_4)\,s'_6c'}$$

$$=\frac{(c_1c_3-s_2s_4d_1d_3)\,c_5}{(c_2c_3-s_1s'_4d_2d_3)\,c'_6}=\frac{(c_2c_3-s_1s_4d_2d_3)\,c_6}{(c_1c-s_2s'_4d_1d_3)\,c'_5}$$

$$=\frac{(c_1c_2+s_3s_4d_1d_2)\,d_3d_6}{(d_1d_2-k^2s_3s_4'c_1c_2)\,c'_5c'_6}=\quad\&c.$$

§ 6. Professor H. J. S. Smith's expression for Ω referred to in § 1 is*

$$\frac{1+\operatorname{dn}2a\,[\operatorname{dn}u]-\operatorname{cn}2a\,[\operatorname{cn}u]-\operatorname{sn}2a\,[\operatorname{sn}u]}{1+\operatorname{dn}2a\,[\operatorname{dn}u]-\operatorname{cn}2a\,[\operatorname{cn}u]+\operatorname{sn}2a\,[\operatorname{sn}u]},$$

where $[\operatorname{dn}u]$, $[\operatorname{cn}u]$, $[\operatorname{sn}u]$ denote respectively

$$\frac{1}{k'^2}\operatorname{dn}u\operatorname{dn}v\operatorname{dn}(u+v),$$

$$\frac{k^2}{k'^2}\operatorname{cn}u\operatorname{cn}v\operatorname{cn}(u+v),$$

$$k^2\operatorname{sn}u\operatorname{sn}v\operatorname{sn}(u+v).$$

If the suffixes 7, 8 refer to the arguments $u+v$, $2a$ respectively, this expression for Ω may be written

$$\frac{k'^2-k^2k'^2s_1s_2s_7s_8-k^2c_1c_2c_7c_8+d_1d_2d_7d_8}{k'^2+k^2k'^2s_1s_2s_7s_8-k^2c_1c_2c_7c_8+d_1d_2d_7d_8}.$$

The eight suffixes are connected with the arguments according to the scheme

3	a	7	$u+v$
4	$u+v-a$	8	$2a$
5	$u+a$	2	v
6	$v+a$	1	u

and it is to be observed that if the first four arguments, *i.e.* those corresponding to the suffixes 3, 4, 5, 6 be denoted by α, β, γ, δ then the second four, *i.e.* those corresponding to the suffixes 7, 8, 2, 1 will be denoted by α', β', γ', δ' where

$$\alpha'=\sigma-\alpha,\quad \beta'=\sigma-\beta,\quad \gamma'=\sigma-\gamma,\quad \delta'=\sigma-\delta.$$

and

$$\sigma=\tfrac{1}{2}(\alpha+\beta+\gamma+\delta)=u+v+a.$$

§ 7. In the *Proceedings of the London Mathematical Society*, t. x. pp. 91–97, Professor Smith has shown that there are eleven cases of the general formula for the multiplication of four theta functions, and has given the corresponding eleven formulæ: these connect the theta functions of any four arguments α, β, γ, δ with those of α', β', γ', δ' where $\alpha'=\sigma-\alpha$, &c. as above. Replacing the theta functions by the elliptic functions sn, cn, dn, and taking the eight arguments α, β, γ, δ, α', β', γ', δ' to be the expressions corresponding to the eight suffixes 3, 4, 5, 6, 7, 8, 2, 1 respectively

* *Proceedings of the London Mathematical Society*, t. i. part viii. p. 13.

in the scheme given in the last section, these eleven formulæ may be written

$$\begin{aligned}
k^2k'^2s_3s_4s_5s_6 &= \tfrac{1}{2}P(\quad k'^2 \quad + k^2k'^2s_1s_2s_7s_8 + k^2c_1c_2c_7c_8 - d_1d_2d_7d_8),\\
k^2c_3c_4c_5c_6 &= \tfrac{1}{2}P(-k'^2 \quad + k^2k'^2s_1s_2s_7s_8 + k^2c_1c_2c_7c_8 + d_1d_2d_7d_8),\\
k'^2 \quad &= \tfrac{1}{2}P(\quad k'^2 \quad + k^2k'^2s_1s_2s_7s_8 - k^2c_1c_2c_7c_8 + d_1d_2d_7d_8),\\
d_3d_4d_5d_6 &= \tfrac{1}{2}P(\quad k'^2 \quad - k^2k'^2s_1s_2s_7s_8 + k^2c_1c_2c_7c_8 + d_1d_2d_7d_8),\\
k'^2s_3s_4 \quad &= \tfrac{1}{2}P(\quad k'^2s_1s_2 + \quad k'^2s_7s_8 \quad - \quad c_1c_2d_7d_8 + c_7c_8d_1d_2),\\
c_3c_4d_5d_6 &= \tfrac{1}{2}P(-k'^2s_1s_2 + \quad k'^2s_7s_8 \quad + \quad c_1c_2d_7d_8 + c_7c_8d_1d_2),\\
c_3c_4 \quad &= \tfrac{1}{2}P(\quad c_1c_2 + \quad c_7c_8 \quad - \quad s_1s_2d_7d_8 + s_7s_8d_1d_2),\\
s_3s_4d_5d_6 &= \tfrac{1}{2}P(- \quad c_1c_2 + \quad c_7c_8 \quad + \quad s_1s_2d_7d_8 + s_7s_8d_1d_2),\\
d_3d_4 \quad &= \tfrac{1}{2}P(\quad d_1d_2 + \quad d_7d_8 \quad + k^2s_1s_2c_7c_8 - k^2s_7s_8c_1c_2),\\
k^2s_3s_4c_5c_6 &= \tfrac{1}{2}P(- \quad d_1d_2 + \quad d_7d_8 \quad + k^2s_1s_2c_7c_8 + k^2s_7s_8c_1c_2),\\
s_3c_4d_6 \quad &= \tfrac{1}{2}P(\quad s_1c_2d_7 + \quad s_2c_1d_8 \quad - \quad s_7c_8d_1 \quad + \quad s_8c_7d_2),
\end{aligned}$$

the value of P being

$$\frac{(1-k^2x^2w^2)(1-k^2y^2z^2)}{(1-k^2x^2z^2)(1-k^2y^2w^2)},$$

where $x = \operatorname{sn}\{\frac{1}{2}(u+v)-a\}$, $w = \operatorname{sn}\{\frac{1}{2}(u+v)+a\}$,
$y = \operatorname{sn}\frac{1}{2}(u-v)$, $z = \operatorname{sn}\frac{1}{2}(u+v)$.

If P_1 denote the value of P when the sign of a is changed, then the ratio $P : P_1$ is equal to the expression (2) in § 1, and therefore we have

$$\frac{P}{P_1} = \Omega.$$

The third of the foregoing eleven formulæ gives Professor Smith's value of Ω, and it will be noticed that this is the only one of the series which does not involve any of the suffixes 3, 4, 5, 6.

The suffixes 3, 4, 5, 6 may be permuted in any manner if at the same time the corresponding suffixes 7, 8, 2, 1 be similarly permuted. The value of P is not affected by any such permutations, as is evident from each of the first four formulæ.

§ 8. Three of the eleven formulæ (viz. the fifth, seventh, and ninth) lead to values of Ω, which do not involve the suffixes 3, 5, 6; we thus find

$$\Omega = \frac{(k'^2 s_1 s_2 - k'^2 s_7 s_8 - c_1 c_2 d_7 d_8 + c_7 c_8 d_1 d_2)\, s_4}{(k'^2 s_1 s_2 + k'^2 s_7 s_8 - c_1 c_2 d_7 d_8 + c_7 c_8 d_1 d_2)\, s_4'}$$
$$= \frac{(c_1 c_2 + c_7 c_8 - s_1 s_2 d_7 d_8 - s_7 s_8 d_1 d_2)\, c_4}{(c_1 c_2 + c_7 c_8 - s_1 s_2 d_7 d_8 + s_7 s_8 d_1 d_2)\, c_4'}$$
$$= \frac{(d_1 d_2 + d_7 d_8 + k^2 s_1 s_2 c_7 c_8 + k^2 s_7 s_8 c_1 c_2)\, d_4}{(d_1 d_2 + d_7 d_8 + k^2 s_1 s_2 c_7 c_8 - k^2 s_7 s_8 c_1 c_2)\, d_4'}.$$

By combining the first four formulæ we have

$$1 + k^2 s_3 s_4 s_5 s_6 = P\,(1 + k^2 s_1 s_2 s_7 s_8),$$
$$k'^2 + d_3 d_4 d_5 d_6 = P\,(k'^2 + d_1 d_2 d_7 d_8),$$
$$k'^2 - k^2 c_3 c_4 c_5 c_6 = P\,(k'^2 - k^2 c_1 c_2 c_7 c_8),$$

and from these equations we deduce that

$$\Omega = \frac{(1 + k^2 s_3 s_4 s_5 s_6)\,(1 - k^2 s_1 s_2 s_7 s_8)}{(1 - k^2 s_3 s_4' s_5' s_6')\,(1 + k^2 s_1 s_2 s_7 s_8)}$$
$$= \frac{k'^2 + d_3 d_4 d_5 d_6}{k'^2 + d_3 d_4' d_5' d_6'}$$
$$= \frac{k'^2 - k^2 c_3 c_4 c_5 c_6}{k'^2 - k^2 c_3 c_4' c_5' c_6'}.$$

A great number of symmetrical expressions for Ω, in which the numerator and denominator differ only by a change of sign of a may be deduced from the eleven formulæ and their permutations; and of course the number of unsymmetrical expressions is much greater. I do not enter upon the discussion of these expressions which, besides being more numerous than those in § 2, involve more arguments.

§ 9. Numerous identical formulæ may be obtained by equating the different expressions for Ω, for example, by combining the three last written expressions for Ω with (i), (ii), (iii) of § 2, we obtain the equations

$$\frac{1 - k^2 s_1 s_2 s_3 s_4}{1 + k^2 s_1 s_2 s_3 s_4'} = \frac{(1 + k^2 s_3 s_4 s_5 s_6)\,(1 - k^2 s_1 s_2 s_7 s_8)}{(1 - k^2 s_3 s_4' s_5' s_6')\,(1 + k^2 s_1 s_2 s_7 s_8)}$$
$$= \frac{(k'^2 - d_1 d_2 d_3 d_4)\, c_5 c_6}{(k'^2 - d_1 d_2 d_3 d_4')\, c_5' c_6'} = \frac{k'^2 + d_3 d_4 d_5 d_6}{k'^2 + d_3 d_4' d_5' d_6'}$$
$$= \frac{(k'^2 + k^2 c_1 c_2 c_3 c_4)\, d_5 d_6}{(k'^2 + k^2 c_1 c_2 c_3 c_4')\, d_5' d_6'} = \frac{k'^2 - k^2 c_3 c_4 c_5 c_6}{k'^2 - k^2 c_3 c_4' c_5' c_6'}.$$

If as a particular case we put $u=v=a$, this group of equations becomes

$$\frac{1-k^2\,\mathrm{sn}^4a}{1+k^2\,\mathrm{sn}^2a\,\mathrm{sn}3a} = 1-k^2\mathrm{sn}^2a\,\mathrm{sn}^2 2a$$

$$= \frac{(k'^2-\mathrm{dn}^4a)\,\mathrm{cn}^2 2a}{k'^2-\mathrm{dn}^2a\;\mathrm{dn}3a} = \frac{k'^2+\mathrm{dn}^2a\,\mathrm{dn}^2 2a}{k'^2+\mathrm{dn}a\;\mathrm{dn}3a}$$

$$= \frac{(k'^2+k^2\,\mathrm{cn}^4a)\,\mathrm{dn}^2 2a}{k'^2+k^2\,\mathrm{cn}^2a\;\mathrm{cn}3a} = \frac{k'^2-k^2\,\mathrm{cn}^2a\,\mathrm{cn}^2 2a}{k'^2-k^2\,\mathrm{cn}a\;\mathrm{cn}3a},$$

from which we may deduce that

$$k'^2+\;\;\mathrm{dn}a\,\mathrm{dn}3a = \frac{k'^2+\mathrm{dn}^2a\,\mathrm{dn}^2 2a}{1-k^2\,\mathrm{sn}^2a\,\mathrm{sn}^2 2a},$$

$$k'^2-k^2\,\mathrm{cn}a\,\mathrm{cn}3a = \frac{k'^2-k^2\,\mathrm{cn}^2a\,\mathrm{cn}^2 2a}{1-k^2\,\mathrm{sn}^2a\,\mathrm{sn}^2 2a},$$

$$k^2\,\mathrm{cn}^2 2a = \frac{-k'^2+\mathrm{dn}^2a\,\mathrm{dn}3a}{1+k^2\,\mathrm{sn}^2a\,\mathrm{sn}3a},$$

$$\mathrm{dn}^2 2a = \frac{k'^2+k^2\,\mathrm{cn}^2a\,\mathrm{cn}3a}{1+k^2\,\mathrm{sn}^2a\,\mathrm{sn}3a}.$$

§10. If we put $k=0$ the expressions for Ω all reduce to unity; but if we put $k=1$ and replace the arguments u, v, &c. by iu, iv, &c., then in virtue of the formulæ

$$\left.\begin{aligned} i\tan u &= \mathrm{sn}\,iu \\ \sec u &= \mathrm{cn}\,iu = \mathrm{dn}\,iu \end{aligned}\right\}(\mathrm{mod.}=1),$$

we obtain from the list in §2 a system of trigonometrical expressions which are all equal to another, viz.

$$\frac{1-\tan u\tan v\tan a\tan(u+v-a)}{1+\tan u\tan v\tan a\tan(u+v+a)}$$

$$= \frac{\cos(u-a)\cos(v-a)\cos(u+v+a)}{\cos(u+a)\cos(v+a)\cos(u+v-a)}$$

$$= \frac{\{\tan u\tan v-\tan a\tan(u+v-a)\}\tan(u+a)\tan(v+a)}{\{\tan u\tan v-\tan u\tan(u+v+a)\}\tan(u-a)\tan(v-a)}$$

$$= \&c.$$

ON THE POTENTIAL DUE TO AN ELECTRIC CURRENT IN AN ELLIPTIC CIRCUIT.

By *W. D. Niven, M.A.*

I. In the *Proceedings of the London Mathematical Society*, Vol. VI., p. 38, Professor Cayley has found the potential due to an elliptic disk of unit density in the following form: If a, b be the semi-axes of the disk, and (x, y, z) any point in space referred to the axes of inertia at the centre as coordinate axes then the potential at the point is

$$2ab\int_\epsilon^\infty dt\sqrt{\left(1-\frac{x^2}{t+a^2}-\frac{y^2}{t+b^2}-\frac{z^2}{t}\right)}\frac{1}{\sqrt{\{t(t+a^2)(t+b^2)\}}},$$

where
$$1-\frac{x^2}{\epsilon+a^2}-\frac{y^2}{\epsilon+b^2}-\frac{z^2}{\epsilon}=0.$$

From this we deduce at once the potential due to unit current circulating in the boundary of the disk by merely differentiating in regard to z, and changing the sign, viz., we have if V be the potential

$$V=2abz\int_\epsilon^\infty \frac{dt}{t\sqrt{\left\{\left(1-\frac{x^2}{t+a^2}-\frac{y^2}{t+b^2}-\frac{z^2}{t}\right)t(t+a^2)(t+b^2)\right\}}}$$

$$=2abz\int_\epsilon^\infty \frac{dt}{t\sqrt{(t-\epsilon)(t-\nu)(t-\nu')}},$$

where ϵ, ν, ν' are the ellipsoidal coordinates of (x, y, z). (See Ferrers's *Spherical Harmonics*, Chap. VI.). If we put $\epsilon, \nu, \nu' = \lambda^2, -p^2, -q^2$, then λ ranges in value from 0 to ∞, p from 0 to b, q from b to a. In the above integral let us write $t-\epsilon=\xi^2$: then

$$V=4abz\int_0^\infty \frac{d\xi}{(\xi^2+\lambda^2)\sqrt{\{(\xi^2+\lambda^2+p^2)(\xi^2+\lambda^2+q^2)\}}}.$$

We now put $\xi=\sqrt{(\lambda^2+p^2)}\tan\theta$, and get

$$V=4abz\int_0^{\frac{1}{2}\pi}\frac{\cos^2\theta\,d\theta}{(\lambda^2+p^2\sin^2\theta)\sqrt{(\lambda^2+q^2\cos^2\theta+p^2\sin^2\theta)}}.$$

This result may be expressed in terms of the first and third complete elliptic integrals, viz. we have

$$\frac{4abz}{\sqrt{(\lambda^2+q^2)}}\left[\frac{\lambda^2+p^2}{\lambda^2 p^2}\,\Pi\left\{\frac{p^2}{\lambda^2},\ \sqrt{\left(\frac{q^2-p^2}{q^2+\lambda^2}\right)}\right\}-\frac{1}{p^2}\,F\left\{\sqrt{\left(\frac{q^2-p^2}{q^2+\lambda^2}\right)}\right\}\right].$$

In the particular case when the ellipse becomes a circle, viz., when $b=a$, we have also $q=a$.

The merit and peculiarity of this expression consist in this, that it is true for all points outside of the current, an advantage not possessed by the series obtained by other methods.

It is doubtful whether the mathematical expression for the potential of an elliptic current is of any great practical value, inasmuch as the mutual induction between two such currents can be expressed only in terms of very complicated integrals. Nevertheless, as the action of such a current on a magnetic needle is determinate, I shall now show how that action may be otherwise calculated.

II. Taking, as before, an ellipse with its axes $2a$, $2b$ in the coordinate axes x, y, then if r be the distance of a point (x, y, z) from the origin, we may easily show that the potential at (x, y, z), due to a distribution of matter of surface density unity on the ellipse, is

$$ab\int_0^{2\pi}\int_0^1 e^{\theta\left(jb\cos\psi\frac{d}{dz}-c\sin\psi\frac{d}{dx}\right)}\frac{1}{r}\,\theta d\theta d\psi\,,$$

where $j=\sqrt{(-1)}$, $c^2=a^2-b^2$ and (x, y, z) is supposed to be so distant that the series obtained by the expansion of this expression is convergent.

If we move the origin to a point at distance v on the negative side of the axis of z the potential becomes

$$ab\int_0^{2\pi}\int_0^1 e^{\theta\left(jb\cos\psi\frac{d}{dz}-c\sin\psi\frac{d}{dx}\right)-v\frac{d}{dz}}\frac{1}{r}\,\theta\,d\theta\,d\psi.$$

Expanding this we find

$$\pi ab\left\{1-v\frac{d}{dz}+(\tfrac{1}{2}v^2-\tfrac{1}{8}b^2)\frac{d^2}{dz^2}+\tfrac{1}{8}c^2\frac{d^2}{dx^2}+\&c.\right\}\frac{1}{r}.$$

Hence the potential at (x, y, z) due to unit current circulating in a wire at the boundary of the ellipse is

$$-\pi ab\left\{\frac{d}{dz}-v\frac{d^2}{dz^2}+(\tfrac{1}{2}v^2-\tfrac{1}{8}b^2)\frac{d^3}{dz^3}+\tfrac{1}{8}c^2\frac{d^2}{dx^2}\frac{d}{dz}+\&c.\right\}\frac{1}{r}.$$

If we write this

$$g_1 \frac{P_1}{r^2} + g_2 \frac{P_2}{r^3} + g_3 \frac{P_3}{r^4} + h_3 \frac{P'_3}{r^4} + \&c.$$

we have

$$g_1 = \pi ab,$$
$$g_2 = 2\pi v ab,$$
$$g_3 = 3\pi (v^2 ab - \tfrac{1}{4} ab^3),$$
$$h_3 = \tfrac{3}{4}\pi ab (a^2 - b^2).$$

III. The expansion just found is true in a certain region remote from the origin. It will be seen presently that the form of the expanded potential at points in the vicinity of the origin is of a totally different form. Within what limits the one expansion or the other may be employed is uncertain. If we suppose that the regions in which the one or other is true are bounded by spherical surfaces whose centre is the origin, then in the region between those two surfaces the proper expansion of the potential must contain harmonics of positive and negative degrees. If, however, there were any doubt as to the right form to use we might always fall back on the expression for V derived from Professor Cayley's formula.

IV. The potential at points near the origin may be discussed as follows:—The practical case will be this: Let the centre of a small magnet NS (fig. 8) be at distance c from the centre on the axis of an elliptic circuit, and let the coordinates of N be ξ, η, $c+\zeta$; then we have to find the potentials at N and S expanded in ascending powers of ξ, η, ζ, the length NS being so small that this expansion is appropriate. The potential at N is given by

$$\Omega = \iint \frac{(c+\zeta)\, dx\, dy}{\{(x-\xi)^2 + (y-\eta)^2 + (c+\zeta)^2\}^{\frac{3}{2}}},$$

the integrations being taken over the area of the ellipse. This expression is derived from the consideration that if P be the position of the element $dx\,dy$, then $NP^3 d\Omega$, where $d\Omega$ is the element of solid angle subtended at N, is equal to

$$dx\,dy \times (c+\zeta).$$

Or, in polar coordinates,

$$\Omega = \iint \frac{(c+\zeta)\, r dr d\theta}{\{r^2 + c^2 - 2(\xi\cos\theta + \eta\sin\theta . r - c\zeta) + \rho^2\}^{\frac{1}{2}}}$$
$$= \iint (Q_0 + Q_1 + Q_2 + Q_3 + \&c.)\, r dr d\theta.$$

where, if we write $r^2 + c^2 = u^{-2}$,

$$Q_0 = cu^3,$$
$$Q_1 = \zeta u^5 + 3c(\xi\cos\theta + \eta\sin\theta . r - c\zeta) u^5,$$
$$Q_2 = 3(\xi\cos\theta + \eta\sin\theta . r - c\zeta)\zeta u^5 - \tfrac{3}{2}c(\xi^2 + \eta^2 + \zeta^2) u^5 + \tfrac{15}{2}c(\xi\cos\theta + \eta\sin\theta . r - c\zeta)^2 u^7,$$
$$Q_3 = -\tfrac{3}{2}\zeta\rho^2 u^5 + \tfrac{15}{2}\zeta\ (\xi\cos\theta + \eta\sin\theta . r - c\zeta)^2 u^7 - \tfrac{15}{2}c\rho^2(\xi\cos\theta + \eta\sin\theta . r - c\zeta) u^7 + \tfrac{35}{2}c\ (\xi\cos\theta + \eta\sin\theta . r - c\zeta)^3 u^9.$$

The integration with regard to r is easily performed. Understanding that r then stands for $ab \div \sqrt{(b^2\cos^2\theta + a^2\sin^2\theta)}$, and that $u^{-2} = r^2 + c^2$, we find for Ω the series $H_0 + H_1 + \&c.$ where

$$H_0 = \int_0^{2\pi} (1 - cu)\, d\theta,$$

$$H_1 = -\zeta \int_0^{2\pi} r^3 u^3 d\theta,$$

$$H_2 = \int_0^{2\pi} (c\zeta^2 + \tfrac{1}{2}c\rho^2 - \tfrac{5}{2}c . \xi^2\cos^2\theta + \eta^2\sin^2\theta)(u^3 - c^{-3})\, d\theta + \tfrac{3}{2}c^3 \int_0^{2\pi} (\xi^2\cos^2\theta + \eta^2\sin^2\theta - \zeta^2)(u^5 - c^{-5})\, d\theta,$$

$$H_3 = \int_0^{2\pi} (\tfrac{1}{2}\zeta\rho^2 - \tfrac{5}{2}\zeta . \xi^2\cos^2\theta + \eta^2\sin^2\theta)(u^3 - c^{-3})\, d\theta + \tfrac{3}{2}c^2\zeta \int_0^{2\pi} \{8(\xi^2\cos^2\theta + \eta^2\sin^2\theta) - \zeta^2 - \rho^2\}(u^5 - c^{-5})\, d\theta + \tfrac{5}{2}c^4\zeta \int_0^{2\pi} \{3(\xi^2\cos^2\theta + \eta^2\sin^2\theta) - \zeta^2\}(u^7 - c^{-7})\, d\theta.$$

It will be observed that the several terms of Ω are solid spherical harmonics of degrees 0, 1, 2, 3, &c., since they each satisfy Laplace's equation. The integrals occurring in the expressions would probably be found most easily by expansions.

As an example, let us now suppose that the circuit is placed in the magnetic meridian with the axis $2a$ horizontal, then the potential of the north pole of the needle will be

$$m(H_0 + H_1 + H_2 + H_3 + \&c.),$$

and of the south

$$-m(H_0 - H_1 + H_2 - H_3 + \&c.),$$

so that the potential energy of the needle in regard to the current γ, is

$$2m\gamma(H_1 + H_3).$$

If the needle is in equilibrium under the earth's magnetism H and the current γ,

$$mH.2l\cos\phi = 2m\gamma\left(\frac{dH_1}{d\phi} + \frac{dH_3}{d\phi}\right),$$

where $\frac{1}{2}\pi - \phi$ is the angle the direction of the needle makes with the magnetic north. Recollecting that $\zeta = \rho\cos\phi$, $= \rho\mu$ suppose, we find

$$\cot\phi = \frac{\gamma}{H}\left(\frac{dH_1}{d\mu} + \&c.\right).$$

In Helmholtz's galvanometer with circular coils, the distance $2c$ between the coils is so chosen with reference to their mean radius, that the coefficient of μ^2 in $\frac{dH_3}{d\mu}$ is zero. If we employ the same consideration here, we find that

$$\int_0^{2\pi}\left\{5\cos^2\theta u^3 - (3 + 24\cos^2\theta)c^2u^5 + (5 + 15\cos^2\theta)c^4u^7\right\}d\theta = 0.$$

When u is constant, this gives the result for the circular coils, viz. that the distance between them is equal to their mean radius.

The ratio $b : a$ is still at our disposal. We may determine it so that the effect on the needle may be a maximum for a given value of a. This gives

$$\int_0^{2\pi}\sin^2\theta u^3 d\theta = 3\int_0^{2\pi}\sin^2\theta c^2u^5 d\theta.$$

The solution of the last two equations would give b and c in terms of a, for the best form of a two coil tangent galvanometer with elliptic windings. The question can hardly however be considered a practical one on account of the

mechanical difficulties of construction and adjustment, whereby greater errors would probably be introduced than would be corrected.

V. The mode of expansion pursued in § IV is not restricted to any particular values of c, and the result may be considered as the expansion in harmonics of positive degree with any point of the axes of the circuit for origin, just as the result in § III is the expansion in harmonics of negative degree. If the centre of the circuit be taken for origin, we have to put $v=0$ in the expressions of § III, and $c=0$ in those of § IV. We thus find for the two forms of the potential

$$\pi ab \left\{ \frac{P_1}{\rho^2} - \tfrac{3}{4}b^2 \frac{P_3}{\rho^4} + \tfrac{3}{4}ab\,(a^2-b^2)\,\frac{P_3'}{\rho^4} + \&\text{c.} \right\},$$

and

$$H_0 + H_1 + H_2 + H_3 + \&\text{c.},$$

where

$$H_0 = 2\pi,$$

$$H_1 = -\zeta \int_0^{2\pi} r^{-1}\, d\theta,$$

$$H_2 = 0,$$

$$H_3 = \int_0^{2\pi} (\tfrac{1}{2}\zeta\rho^2 - \tfrac{5}{2}\zeta.\xi^2 \cos^2\theta + \eta^2 \sin^2\theta)\, r^{-3}\, d\theta,$$

where

$$r^2 = a^2b^2 \div (b^2\cos^2\theta + a^2 \sin^2\theta).$$

As was explained in § III, the value of ρ may be such that neither of these formulæ can be used. We must then fall back on the method of § 1.

I take this opportunity of correcting an error into which I fell in my paper on a special form of Laplace's equation in the last number of the *Messenger*. I there stated (p. 116) that the equation

$$V = A\,\frac{\sin\frac{1}{2}(\psi+\alpha)}{\sqrt{(rr')}}$$

solves the problem of the electrical distribution of a freely charged conductor in the form of a ring. Since however the electrostatic force is discontinuous at the plane containing the axis of the ring, this statement will not hold.

MATHEMATICAL NOTES.

By *R. F. Scott, M.A.*

I. In a paper "Note on a theorem of Prof. Cayley's," *Messenger* VIII. p. 155, it was shown that

$$\begin{vmatrix} \sin(A+F)\sin(B+F)\sin(C+F), & \cos F, & \sin F \\ \sin(A+G)\sin(B+G)\sin(C+G), & \cos G, & \sin G \\ \sin(A+H)\sin(B+H)\sin(C+H), & \cos H, & \sin H \end{vmatrix}$$

$$= \sin(G-H)\sin(H-F)\sin(F-G)\sin(A+B+C+F+G+H).$$

I have recently noticed that this theorem can be both slightly altered and generalised.

Namely, since

$$4\sin a\sin b\sin c = \sin(-a+b+c)+\sin(a-b+c)+\sin(a+b-c)-\sin(a+b+c),$$

if we substitute $A+F$, $B+F$, $C+F$ for a, b, c, we see that the first three terms give a linear function of $\sin F$ and $\cos F$, whose coefficients will be the same as those of similar linear functions of $\sin G$ and $\cos G$ and $\sin H$ and $\cos H$ in the second and third rows respectively. Hence, multiplying the second and third columns of the determinant by the proper numbers and subtracting them from the first, the elements of the first column become, if, for brevity, we write $X = A+B+C$,

$$-\tfrac{1}{4}\sin(X+3F),\ -\tfrac{1}{4}\sin(X+3G),\ -\tfrac{1}{4}\sin(X+3H).$$

Hence we may state the original theorem as follows:

$$\begin{vmatrix} \sin(X+3F), & \cos F, & \sin F \\ \sin(X+3G), & \cos G, & \sin G \\ \sin(X+3H), & \cos H, & \sin H \end{vmatrix}$$

$$= -4\sin(G-H)\sin(H-F)\sin(F-G)\sin(X+F+G+H).$$

In this form the theorem suggests its generalisation. The theorem differs in form for odd and even-ordered determinants. We have

$$\begin{vmatrix} \sin(X+5F), & \cos 3F, & \sin 3F, & \cos F, & \sin F \\ \sin(X+5G), & \cos 3G, & \sin 3G, & \cos G, & \sin G \\ \sin(X+5H), & \cos 3H, & \sin 3H, & \cos H, & \sin H \\ \sin(X+5K), & \cos 3K, & \sin 3K, & \cos K, & \sin K \\ \sin(X+5L), & \cos 3L, & \cos 3L, & \cos L, & \sin L \end{vmatrix}$$

equal to $\sin(X+F+G+H+K+L)$ multiplied by the sines of the differences of F, G, ... and a numerical factor.

The theorem for any odd order can now be written down at once. Instead of X we may write $A+B+C+D+E$ on the right, and replace the elements of the first column by terms such as

$$\sin(A+F)\sin(B+F)\ldots\sin(E+F),$$

the only effect being to alter the numerical factor on the right.

Starting with the formula

$$8\sin a\sin b\sin c\sin d=\cos(a+b+c+d)-\cos(-a+b+c+d)$$
$$-\cos(a-b+c+d)-\cos(a+b-c+d)-\cos(a+b+c-d)$$
$$+\cos(a+b-c-d)+\cos(a-b-c+d)+\cos(a-b+c-d),$$

we see that if we write $A+F$, $B+F$, $C+F$, $D+F$ for a, b, c, d, the terms on the right after the first are linear in $\cos 2F$, $\sin 2F$, 1. Thus, if $X=A+B+C+D$,

$$8\begin{vmatrix}\sin(A+F)\sin(B+F)\sin(C+F)\sin(D+F), & \cos 2F, & \sin 2F, & 1\\ \sin(A+G)\sin(B+G)\sin(C+G)\sin(D+G), & \cos 2G, & \sin 2G, & 1\\ \sin(A+H)\sin(B+H)\sin(C+H)\sin(D+H), & \cos 2H, & \sin 2H, & 1\\ \sin(A+K)\sin(B+K)\sin(C+K)\sin(D+K), & \cos 2K, & \sin 2K, & 1\end{vmatrix}$$

$$=\begin{vmatrix}\cos(X+4F), & \cos 2F, & \sin 2F, & 1\\ \cos(X+4G), & \cos 2G, & \sin 2G, & 1\\ \cos(X+4H), & \cos 2H, & \sin 2H, & 1\\ \cos(X+4K), & \cos 2K, & \sin 2K, & 1\end{vmatrix}.$$

And, by the methods of the paper just quoted, it can be proved that this determinant, up to a numerical factor, is equal to $\sin(X+F+G+H+K)$ multiplied by the sines of the differences between F, ...K.

The theorem for a determinant of order six is that

$$\begin{vmatrix}\cos(X+6F), & \cos 4F, & \sin 4F, & \cos 2F, & \sin 2F, & 1\\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots\\ \cos(X+6M), & \cos 4M, & \sin 4M, & \cos 2M, & \sin 2M, & 1\end{vmatrix}$$

is equal to $\sin(X+F+\ldots+M)$ multiplied by the sines of the differences of F, ...M and a numerical factor. And so on for all determinants of even order.

II. If we differentiate the equation

$$\int_0^1 x^n dx = \frac{1}{n+1}$$

k times with regard to n, we obtain

$$\int_0^1 x^n (\log x)^k dx = \frac{(-1)^k k!}{(n+1)^{k+1}}.$$

Hence we see that

$$\int_0^1 (\log x)^{2n-2} \log\left(\frac{1+x}{1-x}\right) \frac{dx}{x}$$

$$= 2\int_0^1 (\log x)^{2n-2} \left(1 + \frac{x^2}{3} + \frac{x^4}{5} + \ldots\right) dx$$

$$= 2(2n-2)! \left(1 + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \ldots\right)$$

$$= 2(2n-2)! B_{2n-1} \frac{2^{2n-1}\pi^{2n}}{2n!} \left(1 - \frac{1}{2^{2n}}\right)$$

$$= \frac{B_{2n-1}\pi^{2n}(2^{2n}-1)}{2n(2n-1)},$$

where B_{2n-1} is the n^{th} number of Bernoulli.

Similarly

$$\int_0^1 (\log x)^{2n-1} \tan^{-1} x \frac{dx}{x} = -(2n-1)! \left(1 - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \ldots\right)$$

$$= -\frac{E_{2n}\pi^{2n+1}}{n2^{2n+2}},$$

where E_{2n} is the n^{th} number of Euler.

The last theorem may be compared with Prof. Wolstenholme's (Reprints from *Educational Times*), XXX. p. 33).

The integral

$$I = \int_0^\pi \log(1 - 2x\cos\theta + x^2) \log(1 - 2y\cos\theta + y^2)\, d\theta$$

can be found for certain special values of the product xy. For since

$$\log(1 - 2x\cos\theta + x^2) = -2(x\cos\theta + \tfrac{1}{2}x^2\cos 2\theta + \ldots),$$

it follows that the integral

$$= 4\int_0^\pi (xy\cos^2\theta + \frac{x^2y^2}{2^2}\cos^2 2\theta + \frac{x^3y^3}{3^2}\cos^2 3\theta + \ldots)d\theta,$$

all the other terms vanishing

$$= 2\pi\left(xy + \frac{x^2y^2}{2^2} + \frac{x^3y^3}{3^2} + \ldots\right)$$

$$= -2\pi\int_0^{xy} \log(1-u)\frac{du}{u}.$$

This integral is evaluated by Bertrand (*Calcul Intégrale*, Arts. 270 and 271). There it is shewn that if $2xy = 1$,

$$I = \pi\{\tfrac{1}{8}\pi^2 - (\log 2)^2\};$$

and that the integral can also be evaluated when xy satisfies the equation

$$x^2y^2 + xy = 1,$$

viz. that its values are respectively

$$2\pi\{\tfrac{1}{10}\pi^2 - (\log 2 \sin \tfrac{1}{10}\pi)^2\},$$

and

$$2\pi\{\tfrac{1}{10}\pi^2 - (\log 2 \sin \tfrac{1}{10}\pi)^2\}.$$

III. On some determinants of covariants.

If

$$u_1, u_2, \ldots u_p,$$

$$v_1, v_2, \ldots v_p,$$

are two series of binary quantics, and if c_{ik} be the covariant $(12)^r u_i v_k$. Then the determinant

$$|\, c_{ik} \,|$$

is obtained by compounding the two arrays, whose rows are of the form

$$\frac{d^r u_i}{dx^r}, \quad \frac{d^r u_i}{dx^{r-1}dy}, \quad \frac{d^r u_i}{dx^{r-1}dy^2}, \cdots \frac{d^r u_i}{dy^r},$$

$$\frac{d^r v_k}{dy^r}, -r_1\frac{d^r v_k}{dy^{r-1}dx}, r_2\frac{d^r v_k}{dy^{r-2}dx^2}, \cdots \pm\frac{d^r v_k}{dx^r},$$

Hence, the determinant vanishes if $p = r+2$, but if $p = r+1$ it is equal to the product of the two determinants whose i^{th} and k^{th} rows are those above.

Again, with the same notation, the determinant

$$\begin{vmatrix} 0, & v_1, & v_2, & \ldots v_p \\ u_1, & c_{11}, & c_{12}, & \ldots c_{1p} \\ u_2, & c_{21}, & c_{22}, & \ldots c_{2p} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ u_p, & c_{p1}, & c_{p2}, & \ldots c_{pp} \end{vmatrix}$$

is obtained by compounding the arrays

$$x^r, \quad -x^{r-1}y, \quad x^{r-2}y^2, \ldots \pm y^r,$$

$$\frac{d^r u_1}{dy^r}, \quad \frac{d^{r-1}u_1}{dy^{r-1}dx}, \quad \frac{d^r u_1}{dy^{r-2}dx^2}, \ldots \pm \frac{d^r u_1}{dx^r},$$

$$\frac{d^r u_2}{dy^r}, \quad \frac{d^r u_2}{dy^{r-1}dx}, \quad \frac{d^r u_2}{dy^{r-2}dx^2}, \ldots \quad \frac{d^r u_2}{dx^r},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

and

$$y^r, \quad r_1 x y^{r-1}, \quad r_2 x^2 y^{r-2}, \ldots \quad x^r,$$

$$\frac{d^r v_2}{dx^r}, \quad -r_1 \frac{d^r v_1}{dx^{r-1}dy}, \quad r_2 \frac{d^r v_1}{dx^{r-1}dy^2}, \ldots \pm \frac{d^r v_1}{dy^r},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.$$

Hence, this determinant vanishes if $p = r+1$ and is equal to the product of two determinants if $p = r$.

The first of these theorems gives us an identical relation between the covariants, the second an identical relation connecting the covariants and the quantics from which they are formed.

If we take the quantics of order n and put $p = n$ then c_{ik} is the lineo-linear invariant, which we may call I_{ik}. Then the last theorem states that the determinant

$$\begin{vmatrix} 0, & v_1, & v_2, \ldots & v_p \\ u_1, & I_{11}, & I_{12}, \ldots & I_{1p} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ u_p, & I_{p1}, & I_{p2}, \ldots & I_{pp} \end{vmatrix}$$

vanishes if $p = n+1$, but if $p = n$ it is equal to cUV where U and V are two n-tics.

The n-tic U is such that its lineo-linear invariant with any of the quantics $u_1, \ldots u_p$ vanishes.

For if $\quad u_i = a_{in}x^n + n_1 a_{in-1}x^{n-1}y + \ldots + a_{i0}y^n,$

then

$$U = \begin{vmatrix} x^n, & -x^{n-1}y, \ldots \pm y^n \\ a_{10}, & a_{11}, \ldots & a_{1n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{n0}, & a_{n1}, \ldots & a_{nn} \end{vmatrix},$$

and on substituting for $x^n, -x^{n-1}y, \ldots \pm y^n$ the quantities $a_{i0}, a_{i1}, \ldots a_{in}$ (which gives the lineo-linear invariant of u_i and U) this vanishes.

IV. On some exponential symbolic operators.
Let us consider

$$u = e^{h\frac{d^2}{dx\,dy}} e^{kxy}$$

We have $$\frac{du}{dh} = k^2 e^{h\frac{d^2}{dx\,dy}} xy e^{kxy} + ku.$$

$$= k^2 \frac{du}{dk} + ku.$$

The integral of which is

$$ku = \phi\left(\frac{1}{k} - h\right).$$

But when $$h = 0,\ \frac{1}{k}\phi\left(\frac{1}{k}\right) = e^{kxy};$$

therefore $$\phi(p) = \frac{1}{p} e^{\frac{xy}{p}},$$

and $$u = \frac{1}{1-hk} e^{\frac{kxy}{1-hk}}.$$

This method is due to Mr. Crofton (*Reprints from the Educational Times* XIII. p. 51), but the theorem is, I believe, new.

Again, we know that

$$\exp\left(-h\frac{d^2}{dx^2}\right)\exp(-kx^2) = \sqrt{(1+4hk)}\exp\left(\frac{-kx^2}{1+4hk}\right),$$

$$\exp\left(\ \ h\frac{d^2}{dx^2}\right)\exp(-kx^2) = \frac{1}{\sqrt{(1+4hk)}}\exp\left(\frac{-kx^2}{1+4hk}\right),$$

$$\exp\left(\ \ h\frac{d^2}{dx^2}\right)\exp(\ \ kx^2) = \frac{1}{\sqrt{(1-4hk)}}\exp\left(\frac{kx^2}{1-4hk}\right),$$

$$\exp\left(-h\frac{d^2}{dx^2}\right)\exp(\ \ kx^2) = \sqrt{(1+4hk)}\exp\left(\frac{kx^2}{1+4hk}\right).$$

Now let $$u = (a, b, c, f, g, h)(x, y, z)^2$$

be any quadric function, and

$$U = \left(A, B, C, F, G, H\right)\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right)^2,$$

a differential operator formed with the conjugate coefficients. Then we can find the function

$$F = e^{U} e^{u}.$$

For by the same linear transformation we can reduce u, and U to the forms

$$S_1 X^2 + S_2 Y^2 + S_3 Z^2,$$

and

$$S_2 S_3 \frac{d^2}{dX^2} + S_3 S_1 \frac{d^2}{dY^2} + S_1 S_2 \frac{d^2}{dZ^2}.$$

Hence if D is the discriminant of u it follows that

$$F = (1 - 4D)^r \exp\left(\frac{u}{1 - 4D}\right),$$

where $r = \pm\frac{3}{2}$ or $\pm\frac{1}{2}$, according to the signs of the roots of the discriminating cubic.

This theorem can evidently be generalised for any quadric.

Let us further consider the result of operating on the determinant

$$X = \begin{vmatrix} x_1, & x_2 \ldots x_n \\ x_n, & x_1 \ldots x_{n-1} \\ \ldots\ldots\ldots\ldots \\ x_2, & x_3 \ldots x_1 \end{vmatrix},$$

with the symbolic operator

$$\Delta = \begin{vmatrix} \frac{d}{dx_1}, & \frac{d}{dx_2}, & \cdots & \frac{d}{dx_n} \\ \frac{d}{dx_n}, & \frac{d}{dx_1}, & \cdots & \frac{d}{dx_{n-1}} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \frac{d}{dx_2}, & \frac{d}{dx_3}, & \cdots & \frac{d}{dx_1} \end{vmatrix}.$$

We know that X is the product of such factors as

$$y_r = x_1 + w_r x_2 + w_r^2 x_3 + \ldots + w_r^{n-1} x_n.$$

Where $w_1, w_2 \ldots w_n$ are the roots of $w^n = 1$. Hence, if we change the variables from $x_1, x_2 \ldots x_n$ to $y_1, y_2 \ldots y_n$, we see that

$$\frac{d}{dx_r} = w_1^{r-1} \frac{d}{dy_1} + w_2^{r-1} \frac{d}{dy_2} + \ldots + w_n^{r-1} \frac{d}{dy_n}.$$

And hence that

$$\frac{d}{dx_1}+w_r\frac{d}{dx_2}+w_r^2\frac{d}{dx_3}+\ldots+w_r^{n-1}\frac{d}{dx_n}$$

$$=\ldots+\frac{d}{dy_k}(1+w_rw_k+w_r^2w_k^2+\ldots+w_r^{n-1}w_k^{n-1})+\ldots$$

All the terms on the right vanish, except that for which $w_rw_k=1$. Hence we see that

$$n\frac{d}{dy_k}=\frac{d}{dx_1}+w_r\frac{d}{dx_2}+\ldots+w_r^{n-1}\frac{d}{dx_n},$$

and that $$\Delta=n^n\frac{d^n}{dy_1dy_2\ldots dy_n}.$$

Hence $$\Delta X=n^n.$$

I believe the two following theorems are also new.

The ratio of the two determinants

$$\begin{vmatrix} x_1, & x_2, & \ldots x_n \\ x_n, & x_1, & \ldots x_{n-1} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ x_2, & x_3, & \ldots x_1 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 0, & 1, & 1, & \ldots 1 \\ 1, & x_1, & x_2, & \ldots x_n \\ 1, & x_n, & x_1, & \ldots x_{n-1} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ 1, & x_2, & x_3, & \ldots x_1 \end{vmatrix}$$

is $$-\frac{x_1+x_2+\ldots+x_n}{n}.$$

If $x_1, x_2, \ldots x_n$ are the roots of the equation $x^n-1=0$; $y_1, y_2, \ldots y_n$ those of the equation $y^n+1=0$; then if

$$\left\{\begin{array}{cccc} \frac{1}{x_1-y_1}, & \frac{1}{x_1-y_2}, & \ldots & \frac{1}{x_1-y_n} \\ \frac{1}{x_2-y_1}, & \frac{1}{x_2-y_2}, & \ldots & \frac{1}{x_2-y_n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \frac{1}{x_n-y_1}, & \frac{1}{x_n-y_2}, & \ldots & \frac{1}{x_n-y_n} \end{array}\right\}$$

is formed like a determinant, with all its signs positive instead of alternating, it vanishes if n is even, but if n is odd its value is

$$\frac{n(1.3.5\ldots n-2)^2}{2^n}.$$

ON A THEOREM IN STATICS.

By *R. R. Webb, M.A.*

In what follows we shall always premise that the points A, B, C, D determine a finite tetrahedron and that the lengths of the edges BC, CA, AB, AD, BD, CD are a, b, c, a', b', c'. It is further necessary to call to mind the following elementary theorems.

(i). If three forces P, Q, R act in the lines BC, CA, AB of a triangle ABC, then the trilinear equation of the line of action of their resultant is

$$P\alpha + Q\beta + R\gamma = 0,$$

the triangle ABC being the triangle of reference. This result appears at once by noting that the line of action of this resultant is the locus of a point about which the system has no moment, and as the moment about the point $\alpha\beta\gamma$ is represented by the left-hand side of the above-written equation it follows that our assertion is correct. Suppose further that the forces are aP, bQ, cR. Then the above equation is still the line of action of the resultant in question, provided we now consider α, β, γ as the areal coordinates of a point and not trilinears as in the previous case.

(ii). If three forces P', Q', R' act in the lines joining any point D in space to three points A, B, C, then the line of action of this system of forces is in some straight line through D, and meets the plane ABC in a point H, this point is easily found thus; through DA, DH draw a plane meeting BCD in DH', then as DH' is the line of action of the resultant of a force in AD and some force in the plane BCD, it follows that this force in BCD must be in DH', hence, here DH' is the resultant of Q' in BD and R' in CD; on taking moments about H' the condition for this appears to be

$$\frac{Q'}{b'} BH' = \frac{R'}{c'} CH',$$

but the ratio $BH' : CH'$ is that of the areas of the triangles AHB and AHC. Hence

$$\frac{Q'}{b'} : \frac{R'}{c'} :: \text{area } AHC : \text{area } AHB,$$

$$:: \quad \beta \quad : \quad \gamma,$$

where α, β, γ are the areal coordinates of H referred to ABC. So that the line of action of P', Q', R' in AD, BD, CD respectively, is in HD where the areal coordinates of H are given by

$$\alpha : \beta : \gamma :: \frac{P'}{a'} : \frac{Q'}{b'} : \frac{R'}{c'}.$$

If now we replace P' Q', R' by $a'P'$, $b'Q'$, $c'R'$ it follows that the resultant of $a'P'$, $b'Q'$, $c'R'$ in AD, BD, CD respectively, is in HD, where H is the point determined by the areal coordinates α, β, γ, referred to ABC, *or* the tetrahedral coordinates referred to $ABCD$, where

$$\alpha : \beta : \gamma : \delta :: P' : Q' : R' : 0.$$

So that, summarily, from these elementary cases, we see that using tetrahedral coordinates the resultant of aP, bQ, cR in BC, CA, AB respectively, is along the line

$$P\alpha + Q\beta + R\gamma = 0, \quad \delta = 0 \ldots\ldots\ldots\ldots\ldots (1),$$

while that of $a'P'$, $b'Q'$, $c'R'$ in AD, BD, CD is in the line

$$\frac{\alpha}{P'} = \frac{\beta}{Q'} = \frac{\gamma}{R'} \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

tetrahedral coordinates alone being used.

We may remark at once that if the six components reduce to a single resultant the lines (1) and (2) just written down must meet, and therefore

$$PP' + QQ' + RR' = 0.$$

We now pass on to what is really the object of this note, viz. the consideration of (1) the component forces in the six lines BC, CA, AB, AD, BD, CD, which are equivalent to a force in a given line; and (2) the component couples whose axes lie along the same lines which give a resultant couple whose axis coincides with the direction of a given line.

(1) Suppose we consider a force in the line determined in areal coordinates by the planes

$$\left.\begin{aligned} l\alpha + m\beta + n\gamma + r\delta &= 0 \\ l'\alpha + m'\beta + n'\gamma + r'\delta &= 0 \end{aligned}\right\},$$

and for shortness let us designate the six functions

$lr' - l'r$, $mr' - m'r$, $nr' - n'r$, $mn' - m'n$, $nl' - n'l$, $lm' - l'm$

by the six symbols x, y, z, x', y', z' respectively.

The planes through A, B, C, D respectively, and the given line are

$$\left.\begin{aligned} &\beta(ml'-m'l)+\gamma(nl'-n'l)+\delta(rl'-r'l)=0\\ &\gamma(nm'-n'm)+\delta(rm'-r'm)+\alpha(lm'-l'm)=0\\ &\delta(rn'-r'n)+\alpha(ln'-l'n)+\beta(mn'-m'n)=0\\ &\alpha(lr'-l'r)+\beta(mr'-m'r)+\gamma(nr'-n'r)=0 \end{aligned}\right\},$$

respectively, or availing ourselves of the quantities xyz, $x'y'z'$,

$$\left.\begin{aligned} &\quad . \quad +z'\beta-y'\gamma+x\delta=0\\ &-z'\alpha \quad . \quad +x'\gamma+y\delta=0\\ &+y'\alpha-x'\beta \quad . \quad +z\delta=0\\ &+x\alpha+y\beta+z\gamma \quad . \quad =0 \end{aligned}\right\} \ldots\ldots\ldots\ldots (\mathrm{I}).$$

Hence the areal equations to the lines in which these planes meet the respectively opposite faces of the tetrahedron are, when referred to those opposite faces as triangles of reference, represented by the four equations just written down. Suppose now we consider any line through A and the given line; as this line meets the resultant of the components in AB, AC, AD, and also the resultant of the whole set, it must of necessity meet the resultant of the components in BC, CD, DB; but this line is, for the components aP, bQ, cR, $a'P'$, $b'Q'$, $c'R'$,

$$.+R'\beta-Q'\gamma+P\delta=0, \text{ and } \alpha=0,$$

in tetrahedrals; hence the plane through A and the resultant of the six components is

$$.+R'\beta-Q'\gamma+P\delta=0,$$

and hence the four planes

$$\left.\begin{aligned} &\quad . \quad +R'\beta-Q'\gamma+P\delta=0\\ &-R'\alpha \quad . \quad +P'\gamma+Q\delta=0\\ &+Q'\alpha-P'\beta \quad . \quad +R\delta=0\\ &+P\alpha+Q\beta+R\gamma \quad . \quad =0 \end{aligned}\right\} \ldots\ldots\ldots (\mathrm{II}),$$

must be identical with the four planes (I) when the resultant of these components is in the given line.

Hence we conclude by mere inspection of (I) and (II) that

$$\frac{P}{x}=\frac{Q}{y}=\frac{R}{z}=\frac{P'}{x'}=\frac{Q'}{y'}=\frac{R'}{z'},$$

and the six components reducing to a force in the given line bear to each other unique ratios, and in order to find them for unit force we have merely to introduce the additional fact that the resultant has magnitude unity. Hence we infer that unit force in a given line can be replaced by six components along the edges of a tetrahedron in one and only one way. It is, however, not so with couples, as is *à priori* obvious, and we now proceed to investigate the amount of freedom enjoyed by the components in that case.

(2) Our method of procedure is to some extent the counterpart of the former, instead of forming the four planes through A, B, C, D respectively and the given line, we have here need of the four lines through A, B, C, D, each parallel to the given line. The line through A parallel to the given one is clearly the intersection of the planes through A respectively parallel to the planes determining the given line, these are

$$l\alpha + m\beta + n\gamma + r\delta - l\,(\alpha+\beta+\gamma+\delta) = 0,$$
$$l'\alpha + m'\beta + n'\gamma + r'\delta - l'\,(\alpha+\beta+\gamma+\delta) = 0,$$

or
$$\left.\begin{aligned}\beta\,(m-l) + \gamma\,(n-l) + \delta\,(r-l) &= 0\\ \beta'\,(m'-l') + \gamma\,(n'-l') + \delta\,(r'-l') &= 0\end{aligned}\right\}.$$

Hence, solving and making use of our previous notation, this line appears to be

$$\frac{\beta}{x+y'-z} = \frac{\gamma}{-x+y+z'} = \frac{\delta}{-x'-y'-z'}.$$

Hence, in the present instance the four lines in question through A, B, C, D respectively and this given line

are
$$\frac{\beta}{x+y'-z} = \frac{\gamma}{-x+y+z'} = \frac{\delta}{-x'-y'-z'},$$
$$\frac{\alpha}{x'-y+z} = \quad\cdot\quad = \frac{\gamma}{-x+y+z'} = \frac{\delta}{-x'-y'-z'},$$
$$\frac{\alpha}{x'-y+z} = \frac{\beta}{x+y'-z} = \quad\cdot\quad = \frac{\delta}{-x'-y'-z'},$$
$$\frac{\alpha}{x'-y+z} = \frac{\beta}{x+y'-z} = \frac{\gamma}{-x+y+z'} \quad\cdot\quad\cdot$$

Reverting to the second of the theorems, cited at the outset, and remembering that axes of couples conform to the method of composition and resolution therein laid down,

it follows that a couple whose axis is in any one of the above written lines, is equivalent to either of the following systems of components

$$0,\ b(-x+y+z'),\ -c(x+y'-z),\ a'(x'+y'+z'),\ 0,\ 0,$$
$$-a(-x+y+z'),\ 0,\ +c(x'-y+z),\ 0,\ +b'(x'+y'+z'),\ 0,$$
$$+a(x+y'-z),\ -b(x'-y+z),\ 0,\ 0,\ 0,\ +c'(x'+y'+z'),$$
$$0,\ 0,\ 0,\ a'(x'-y+z),\ b'(x+y'-z),\ c'(-x+y+z').$$

Hence as the axis of a couple is merely a direction and a magnitude, it follows that the couple in the given line is equivalent, not merely to the component couples given by any one of the above rows, but also to a component set obtained from the above diagram by multiplying the rows in succession by any quantities whatever and then adding the columns. Let now

$$x'-y+z \equiv L,$$
$$x+y'-z \equiv M,$$
$$-x+y+z' \equiv N,$$
$$-x'-y'-z' \equiv R.$$

Then of course $L+M+N+R=0,$

and the above scheme of component couples is

$$\begin{array}{cccccc} 0, & bN, & -cM, & a'R, & 0, & 0, \\ -aN, & 0, & +cL, & 0, & b'R, & 0, \\ +aM, & -bL, & 0, & 0, & 0, & c'R, \\ 0, & 0, & 0, & -a'L & -b'M, & -c'N. \end{array}$$

On following out the directions for forming a general couple set, and choosing $L'M'N'R'$ as the multipliers wherewith to multiply the successive rows, it follows that the system of component couples is determined by

$$a(MN'-M'N),\ b(NL'-N'L),\ c(LM'-L'M),$$
$$a'(RL'-R'L),\ b'(RM'-R'M),\ c'(RN'-R'N).$$

Now

$$a(LR'-L'R),\ b(MR'-M'R),\ c(NR'-N'R),$$
$$a'(MN'-M'N),\ b'(NL'-N'L),\ c'(LM'-L'M),$$

are the six component forces in BC, CA, AB, AD, BD, CD that lead to a single resultant in the line

$$\left.\begin{aligned} L\alpha + M\beta + N\gamma + R\delta = 0 \\ L'\alpha + M'\beta + N'\gamma + R'\delta = 0 \end{aligned}\right\}.$$

Hence we infer that if any force in the line just written down be equivalent to the six components

$$aX,\ bY,\ cZ,\ a'X',\ b'Y',\ c'Z',$$

then any couple whose axis is along the line

$$l\alpha + m\beta + n\gamma + r\delta = 0,$$
$$l'\alpha + m'\beta + n'\gamma + r'\delta = 0,$$

is equivalent to the six component couples determined by

$$aX',\ bY',\ cZ',\ -a'X,\ -b'Y,\ -c'Z.$$

Further, it follows that the system just written down is the most general admissible, for in order to determine the six component couples we require six equations, and the expression of the equivalence of six components taken at random to the given couple will reduce them to three, and these, being perfectly arbitrary appear in the foregoing theorem as the mutual ratios of the four arbitrary constants $L'M'N'R'$. Hence, summarily, it appears that the following theorem results:

A couple whose axis is in

$$l\alpha + m\beta + n\gamma + r\delta = 0,$$
$$l'\alpha + m'\beta + n'\gamma + r'\delta = 0.$$

is equivalent to six component couples whose axes lie along the edges BC, CA, AB, AD, BD, CD, whose moments are severally proportional to

$$aX',\ bY',\ cZ',\ -a'X,\ -b'Y,\ -c'Z,$$

where $aX,\ bY,\ cZ,\ a'X',\ b'Y',\ c'Z'$,

are the components in BC, CA, AB, AD, BD, CD of any force in any line in the plane, whose equation is

$$L\alpha + M\beta + N\gamma + R\delta = 0,$$

where

$$L = x' - y + z,$$
$$M = x + y' - z,$$
$$N = -x + y + z',$$
$$R = -x' - y' - z'.$$

This plane, when expanded, is

$$\alpha(x'-y+z)+\beta(x+y'-z)+\gamma(-x+y+z')-\delta(x'+y'+z')=0,$$

or $$x(\beta-\gamma)+y(\gamma-\alpha)+z(\alpha-\beta)$$
$$+x'(\alpha-\delta)+y'(\beta-\delta)+z'(\gamma-\delta)=0,$$

and in this shape is somewhat striking when we remember that unit force in the given line is equivalent to the components

	ax,	by,	cz,	$a'x'$,	$b'y'$,	$c'z'$,
in	BC,	CA,	AB,	AD,	BD,	CD,

respectively.

EXTENSION TO CURVED SURFACES OF A FORMER THEOREM ON PLANE AREAS.

By *E. B. Elliott, M.A.*

THE theorem which I gave in extension of Holditch's as to areas in a plane, (*Messenger*, vol VII., p. 150), admits of a limited application to areas on curved surfaces, which may have interest, although probably little practical value.

The theorem proved was that if (A) and (B) be two closed areas in the same plane whose perimeter are described in the same sense of rotation by the two ends of a varying straight line AB, if (C) be the area enclosed by the locus of a point C dividing AB in a fixed ratio $m:n$, and if S be the area enclosed by the locus of the second end of a straight line drawn from a fixed point always equal and parallel to AB, then

$$(C)=\frac{m(B)+n(A)}{m+n}-\frac{mn}{(m+n)^2}S \quad \text{.....(1).}$$

To extend this, as desired, use has to be made of three facts of elementary geometry, viz.

(i) That A_1, A_2, A_3 and B_1, B_2, B_3 being sets of three (or any number of) points in each of two parallel planes, the points C_1, C_2, C_3 which divide A_1B_1, A_2B_2, A_3B_3 in any the same ratio $m:n$ lie in another parallel plane.

(ii) That, in the same case, if from any fixed point lines be drawn parallel and equal to A_1B_1, A_2B_2, A_3B_3, their other extremities lie in a plane parallel to each of the above planes, and

(iii) That any orthogonal projection of a divided straight line is a similarly divided straight line.

Now first, a like relation to (1) holds if (A) and (B) instead of being in the same are in parallel planes. For the locus of points C dividing AB in the constant ratio $m : n$ by (i), and by (ii) the relative locus of B with regard to A, are plane curves in planes parallel to those of (A) and (B). Call their areas (C) and S respectively. Then projecting the system orthogonally upon any plane and using (iii), the projected areas (A'), (B'), (C'), S' are such as have been proved to be connected by the relation (i). Hence, multiplying by the secant of the angle of projection the areas (A), (B), (C), S obey the same law.

This is in particular true if the parallel areas be infinitesimal ones $\delta(A)$, $\delta(B)$, $\delta(C)$, δS, *i.e.*

$$\delta(C) = \frac{m\delta(B) + n\delta(A)}{m+n} - \frac{mn}{(m+n)^2}\delta S.$$

Now take an indefinite number of contiguous infinitesimal areas $\delta(A)$ making up a portion of curved surface (A), and an indefinite number of parallel elements $\delta(B)$ making up another such portion (B), and let (C) and S be the areas of curved surface made up by the assemblages of elements $\delta(C)$ and δS. Then, summing, the same relation (i) connects these portions of curved surface, and we may state the obtained result thus:

Let (A) and (B) be two continuous portions of curved surfaces such that points A, completely covering the one have corresponding to them points B completely covering the other in such a way that the tangent planes at pairs of corresponding points are parallel, points in particular all round the outer boundaries of the two portions of surface corresponding in this manner; let (C) be the area of the portion of curved surface covered by points dividing AB in the constant ratio $m : n$, and let S be that which would be covered by the free ends of lines drawn from a fixed point parallel and equal to all the lines AB. Then (A), (B), (C), S are connected by the relation (i).

In particular, of course, our given portions of surface, and consequently also the derived ones, might be entire closed surfaces. The interpretation of sign in (C) is however likely often to give trouble.

The application is immediate to areas of similar and similarly situated surfaces; but the result is more easily got directly.

Another class of examples is afforded by parallel surfaces. For instance, we obtain that, given a portion of surface, the sum of the areas of the corresponding portions of the two parallel sheets at a distance R exceeds that of the given portion by the area cut out of a sphere of radius R by radii parallel to the extreme normals; and, in particular, that, given a closed convex surface, the sum of the areas of the two parallel sheets at distance R exceeds its area by $4\pi R^2$.

It is perhaps worth while to call attention to a relation similar to the above, but simpler, connecting lengths of associated curves, which is, however, far too obvious by the method of infinitesimals to have escaped notice often before.

It may be stated: *If* (α), (β) *be the lengths of two curves, which may or may not be closed, in the same plane, in parallel planes, or in space, such that tangents at points all along the one are parallel respectively to those at points all along the other, and if* (γ) *be the length of the curved locus of points dividing connectors of corresponding points in the fixed ratio* $m : n$, *then*

$$(\gamma) = \frac{m(\beta) + n(\alpha)}{m+n}.$$

This is true at least if the intersection of each connector of corresponding points with its consecutive one does not lie between the two curves. In case each is between them, the sum in the numerator on the right must be replaced by a difference. If the envelope of the connectors cross either of the curves, the parts between each two crossings must be taken separately.

A PARADOX IN THE THEORY OF ORDINARY DIFFERENTIAL EQUATIONS.

By *H. W. Lloyd Tanner, M.A.*

1. A SYSTEM of n equations, such as

$$f(y, x, c_1, c_2, \ldots c_m) = 0,$$

wherein $c_1, c_2, \ldots c_m$ are constants, may be regarded as a general solution of a differential equation of the first order, provided that

$$m + 1 = 2n.$$

For from the given system, and the n equations thence deduced by differentiation with respect to x, the $2n-1$ constants may be eliminated, leaving a relation between $\frac{dy}{dx}$, y, x, which is the differential equation in question. It is assumed that both the variables y, x are actually present in each f.

2. Another possible form of solution, which includes the above as a special case, is given by p equations, such as

$$f(y, x, c_1, c_2, \ldots c_{m'}) = 0,$$

and q equations of the form

$$g(c_1, c_2, \ldots c_{m'}) = 0.$$

Here we must have

$$m' + 1 = 2p + q;$$

for the equations from which the constants are to be eliminated consist of the p equations $f=0$, the p equations thence obtained by differentiation, and the q equations $g=0$; and these are, in general, sufficient for the elimination of $2p+q-1$ quantities.

3. Starting with the system of Art. 1, we can, by proper transformations obtain an equivalent system of the form described in Art. 2. In this new system p may have any value not less than 2 nor greater than n; and q becomes $n-p$. Thus, although from the original system we can eliminate $2n-1$ constants by differentiation, from a precisely equivalent system we can only eliminate $n+p-1$ constants. The same result may be otherwise expressed. The system of Art. 2, involving $2p+q-1$ constants, appears as the solution of a single differential equation of the first order, from which the constants have disappeared. From the equivalent system in which x, y have been eliminated from r of the equations, we can obtain differential equations of the first order, but in each of these r of the constants appear. If, on the other hand, we get an equivalent system in which p is increased by r (and therefore q diminished by r), this system appears as the solution common to r differential equations of the first order.

4. The ground of the contradictions thus obtained is the assumption that the parameters c are constant. It is easy to see that this assumption is, at any rate in some cases, unwarranted. Take for instance any two of the equations of Art. 1,

and from them eliminate y; suppose the resulting equation solved for x, so that we have

$$x = f(c_1, c_2, \ldots c_m).$$

Such an equation is incompatible with the assumption that the c's are all constant. It is, however, easy to see that in the process of Art. 2 the assumption is legitimate. For on this assumption we get $2p + q$ equations, from which may be deduced a relation that holds good whatever c_1, c_2,... may be. The assumption ceases to be valid when the system $f = 0$, $g = 0$ is replaced by another with different values of p, q, although the two may be algebraically equivalent.

TRANSACTIONS OF SOCIETIES.

London Mathematical Society.

Thursday, January 13th, 1881.—S. Roberts, Esq., F.R.S., *President*, in the Chair. Messrs. James Parker Smith, M.A., Fellow of Trinity College, Cambridge; Oscar Howard Mitchell, Fellow of Johns Hopkins University, Baltimore; Thomas Craig, United States Coast Survey Office, Washington; and Miss C. A. Scott, Emmanuel House, Cambridge, were elected Members.

The Chairman, with a few introductory remarks, called upon Dr. Hirst to speak upon the loss the Society had sustained by the death of its first Foreign Member, M. Chasles. Dr. Hirst, in feeling terms, rapidly glanced at the career and work of the deceased geometer, and stated some interesting particulars in illustration of the simplicity of character, geniality and kind-heartedness of his old tutor and friend, which were very warmly applauded by the members present. Dr. Henrici, F.R.S., drew attention to a continuation and projected completion, now in progress, of Poggendorff's "Dictionary of the Exact Sciences, Biographical and Literary." The Chairman mentioned that Mr. Howard Elphinstone had presented several valuable mathematical works to the library, and proposed that a vote of thanks should be passed to that gentleman for his gift. The vote was passed unanimously.

The following communications were made: "On an apparently paradoxical relation of the circle, parabola, and hyperbola," Mr. A. J. Ellis, F.R.S.; "A proof of the differential equation which is satisfied by the hypergeometric series," Rev. T. R. Terry, M.A.; "On the periodicity of hyper-elliptic integrals of the first class," Mr. W. R. Westropp Roberts, M.A.; "On the tangents drawn from a point to a nodal cubic," Mr. R. A. Roberts, B.A.; "Sur une propriété du paramètre de la transformée canonique des formes cubiques ternaires," Signor Briocshi (Milan); "Note on a kinematical theorem connected with the rectilinear courses of two uniformly sailing vessels," Mr. C. W. Merrifield, F.R.S.; "A partition-problem connected with a triangle and its successive pedal triangles," Mr. J. W. L. Glaisher, F.R.S.

ON CONICS CIRCUMSCRIBED ABOUT, OR INSCRIBED IN, TRIANGLES WHICH ARE SELF-CONJUGATE WITH RESPECT TO A GIVEN CONIC.

By *A. F. Torry, M.A.*

DEF. (1). A CONIC circumscribed about a triangle self-conjugate with respect to a given conic is said to be *harmonically circumscribed* to the given conic.

DEF. (2). A conic inscribed in a triangle self-conjugate with respect to a given conic is said to be *harmonically inscribed* to the given conic.

DEF. (3). Two straight lines are said to be conjugate with respect to a conic if the pole of either of them is situated on the other.

Conjugate diameters are included in this definition.

DEF. (4). Two points are said to be conjugate with respect to a conic when the polar of one passes through the other.

2. PROP. I. *If a conic S be harmonically circumscribed to another C, the polar with regard to C of any point P on S will meet S again in two conjugate points, which together with P form a triad self-conjugate with respect to C.*

Let S (fig. 9) circumscribe the self-conjugate triad DEF, and from any point P on S let tangents PK, PK' be drawn to C, and let KK' cut S in Q, R, PQR shall be a self-conjugate triad with respect to C.

The conic S, and opposite sides of the quadrilateral $PDEF$ cut any transversal in points in involution. Take KK' the polar of P as the transversal, and let it be cut by PD, EF in N, M, and by PF, DE in N', M'. Then, because M is the pole of PD, $[NKMK']$ is harmonic, similarly $[N'KM'K']$ is harmonic. These conditions determine K, K' to be the foci of the system, and therefore $[RK'QK]$ is also harmonic, PQ is therefore the polar of R, and the triangle PQR is self-conjugate.

COR. 1. An infinite number of conjugate triads can be inscribed in any harmonically circumscribed conic, but when one angular point is given the other two are determined.

Cor. 2. When Q, R coincide at one of the points of intersection of the two conics, QR becomes a tangent to S, and PQ a tangent to C. Hence, where the conics intersect the tangent to S has its pole with respect to C on S. The real and imaginary triangles inscribed in S and self-conjugate with respect to C, are separated by the cases of *flat triangles* whose indefinitely small bases coincide with the points of intersection of C and S.

Ex. If two triangles are self-conjugate with respect to the same conic, their six angular points lie on another conic.

For if a conic be described through D, E, F, P and Q the foregoing construction shews that the other point besides Q in which the conic is cut by the polar of P must be R the pole of PQ.

3. Prop. II. *If two conics are harmonically circumscribed to a given conic, the opposite sides of the quadrilateral formed by joining their points of intersection are conjugate lines with respect to the given conic.*

Let P be one of the four points of intersection, and let its polar be taken as a transversal cut by the two conics and by pairs of opposite sides of the quadrilateral in a series of points in involution.

The two conics being cut in conjugate points, the foci of involution are determined as in the previous proposition to be K, K' where the polar of P cuts the given conic. Hence, if a pair of opposite sides of the quadrilateral cut this transversal in M, N, $[MKNK']$ is a harmonic range, and therefore M, N are the poles of PN, PM, and the proposition follows.

Cor. 1. Pairs of opposite sides of the above quadrilateral are sides of self-conjugate triads, and are limits of conics harmonically circumscribed to the given conic.

Cor. 2. Since four points can be joined two and two in *three* ways by as many pairs of straight lines, the proposition, with Cor. 1., shews that if two pairs of these straight lines are conjugate, the third pair will be so also.

This theorem and its reciprocal are due to Hesse. See Mr. Taylor's new book on *Conics*, page 333, note 1.

Cor. 3. All conics circumscribing such a quadrilateral are harmonically circumscribed to the given conic.

4. PROP. III. *If any harmonically circumscribed conic, or two sides of a conjugate triad, cut a pair of conjugate axes CP, CD in M, M'; N, N', then will*

$$\frac{CP^2}{CM.CM'} + \frac{CD^2}{CN.CN'} = 1.$$

The conjugate diameters form a harmonically circumscribed conic, and therefore by Prop. II, MN, $M'N'$ are conjugate lines.

Let (α, β) the pole of MN lie on $M'N'$,

$$\frac{x\alpha}{a^2} + \frac{y\beta}{b^2} = 1, \quad \frac{x}{CM'} + \frac{y}{CN'} = 1,$$

will be the equations to MN, $M'N'$. Therefore

$$CM = \frac{a^2}{\alpha}, \quad CN = \frac{b^2}{\beta} \text{ and } \frac{\alpha}{CM'} + \frac{\beta}{CN'} = 1;$$

therefore

$$\frac{a^2}{CM.CM'} + \frac{b^2}{CN.CN'} = 1.$$

COR. Conversely, if the above condition be satisfied, MN $M'N'$ are conjugate lines, and the first conic is harmonically circumscribed to the second.

5. *On harmonically inscribed conics.*

The following definitions and theorems are the reciprocals of those relating to harmonically circumscribed conics.

A conic C is harmonically inscribed to another S.

(1) If it touches three sides of a triangle self-conjugate with respect to S. Or,

(2) If from the pole with respect to S of one tangent to C, the two tangents drawn to C are conjugate lines with respect to S. Or,

(3) If, M and N being the respective points of contact of a common tangent to C and S, the other tangent to C from N is the polar of the point M with respect to S.

In this case two of the tangents to C coincide.

Or (4) tangents being drawn to C parallel to conjugate axes Sp, Sd of S, and cutting Sd, Sp in m, m'; n, n', if

$$\frac{Sm.Sm'}{Sp^2} + \frac{Sn.Sn'}{Sd^2} = 1.$$

(5) Conversely, the above conditions are fulfilled in the case of all conics harmonically inscribed to the given conic S.

6. PROP. IV. *If a conic S be harmonically circumscribed to another C, then will C be harmonically inscribed to S.*

Draw MN (fig. 10) a common tangent to the two conics, and NP the second tangent from N to C, and let MP the polar of N with respect to C cut S in QR.

Because S is harmonically circumscribed to C, Q, R are, by Prop. I, conjugate points, and NQR is a conjugate triad with respect to C.

Therefore $[MQPR]$ is a harmonic range, and consequently NP the polar of M with respect to S.

Therefore, by section 5 (3) C is harmonically inscribed to S.

7. *Another proof.* By Prop. I. Cor. 2, S is harmonically circumscribed to C if, PM, PN being tangents to C, S passes through P and touches MN in M. Draw (fig. 11) the tangent to S at P to meet NM in D, and from D draw tangents to C whose chord of contact will pass through P and be cut harmonically by MN and the curve. Therefore, the pencil at D will be harmonic; and if the tangents from D cut PM in E, F the range $[PEMF]$ will be harmonic, DE will be the polar of F with respect to S, and the triangle DEF formed by three tangents to C will be self-conjugate with respect to S. Therefore, C is harmonically inscribed to S.

8. The foregoing result follows analytically from the following:

PROP. V. *If pairs of conjugate axes parallel to one another be taken in two conics S and C, and h, k be the coordinates of the centre of either conic referred to the axes of the other, the condition that S should be harmonically circumscribed to C, or that C should be harmonically inscribed to S is*

$$\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}=\frac{h^2}{\alpha^2}+\frac{k^2}{\beta^2}-1,$$

a, b being the semi-conjugate axes of C, α, β those parallel to them of S.

Referring to the condition of Prop. III., we have by a well-known property of conics

$$\frac{CM.CM'}{\alpha^2}=\frac{CN.CN'}{\beta^2}=\frac{h^2}{\alpha^2}+\frac{k^2}{\beta^2}-1,$$

which reduces the condition to the required form.

Referring again to section 5 (4) we have $Sm.Sm' = a^2 - h^2$ and $Sn.Sn' = b^2 - k^2$, whence also we obtain the same result.

N.B. The directions of the axes used above are known to be those of the parabolas which can be described through the points of intersection of the two conics.

9. Certain particular cases of the foregoing propositions deserve notice either for the simplification of the theorems themselves or of the proofs applicable to them.

ON HARMONICALLY CIRCUMSCRIBED CIRCLES.

PROP. VI. *The tangents from the centre of a conic to all circles harmonically circumscribed to it are equal in length.*

Let DEF (fig. 12) be a self-conjugate triad and let EF the polar of D cut the curve in QQ' and CD in V, let CA be the semi-diameter through D and CB its conjugate, and let the circle circumscribing DEF cut CD again in K.

Now $$\frac{VQ^2}{CB^2} = \frac{CV.VD}{CA^2}.$$

Also $$VQ^2 = VE.VF = VK.VD,$$

and $$CA^2 = CV.CD.$$

Hence $$CB^2 = VK.CD;$$

therefore $$CA^2 + CB^2 = CK.CD.$$

But $CK.CD$ is equal to the square of the tangent from C to the circle, which is therefore of constant length equal to the radius of the director circle.

COR. 1. All harmonically circumscribed circles are orthogonal to the director circle.

COR. 2. The radical axes of all pairs of circles harmonically circumscribed to a given conic pass through the centre of that conic.

COR. 3. If such a circle cut any diameter of the conic in M, M', $CM.CM' = a^2 + b^2$.

COR. 4. If r be the radius of the circle, the distance of its centre from that of the ellipse $= \sqrt{(a^2 + b^2 + r^2)}$.

Remembering that all circles have two imaginary points at infinity in common, we might have deduced all the

propositions we have proved for harmonically circumscribed conics by projection from the corresponding properties of harmonically circumscribed circles.

HARMONICALLY INSCRIBED CIRCLES.

10. The propositions already stated for conics in general apply to circles. The chief simplification occurs when we consider the intercepts upon the principal axes of the conic of tangents of the circle parallel to those axes. Let h, k be the coordinates of the centre of the circle referred to the axes of the ellipse. The condition becomes $\frac{h^2-r^2}{a^2}+\frac{k^2-r^2}{b^2}=1$, r being the radius of the circle, which shews that if a circle of constant radius be harmonically inscribed to a given conic, the locus of its centre is a conic similar to the given one.

11. *A circle harmonically circumscribed to an ellipse and passing through one of the foci will have its centre on the corresponding directrix.*

For if the directrix be the base of a self-conjugate triangle, the vertical angle of the triangle at the focus will be a right angle.

12. PROP. VII. *The centres of circles harmonically circumscribed to a parabola lie on the directrix.*

This a particular case of Prop. VI. Cor. 1, but appears deserving of an independent proof.

Let EF (fig. 13) one of the sides of a self-conjugate triangle DEF cut the parabola in Q, Q', and let the diameter through D meet QQ' in V, the parabola in P, the directrix in M and the circle circumscribing DEF in D'.

Then $VD.VD' = VE.VF = VQ^2$,

but $VQ^2 = 4SP.PV = 2SP.VD$;

therefore $VD' = 2SP = 2PM$.

Also $VD = 2PD$,

hence $DD' = 2DM$, and the directrix bisects DD' at right angles and passes through the centre of the circle.

COR. 1. Conversely, any circle whose centre is on the directrix is harmonically circumscribed to the parabola.

Cor. 2. Since all parabolas touch the line at infinity and the tangents drawn from the centre of the circle to the parabola, being at right angles, are conjugate axes of the circle; it follows that the parabola is inscribed in a triangle self-conjugate to the circle, which proves Prop. IV. for this case.

13. *If any conic be harmonically circumscribed to a parabola the tangents to the parabola from the centre of the conic will be conjugate diameters of the conic.*

Project the conic orthogonally into a circle and the proposition follows because tangents to the parabola which meet on the directrix are at right angles.

Otherwise. A parabola has always one tangent at infinity. If, therefore, it touches conjugate diameters of a conic it is harmonically inscribed in that conic, and, therefore, the conic is harmonically circumscribed to the parabola.

14. *If a conic be harmonically inscribed to a parabola, and its diameter CP, drawn conjugate to the axis of the parabola, cut the parabola in M* and *N, then* $CM.CN = CP^2$.

The parabola is harmonically circumscribed to the conic, and one of the denominators in the general formula becomes infinite when applied to this case.

Cor. If a circle of constant radius be harmonically inscribed to a parabola, the locus of its centre is a parabola having the same axis and parameter as the given one.

If the equation to the given parabola be $y^2 = 4ax$, that of the locus will be $y^2 - r^2 = 4ax$.

16. The following proposition may, in the case of the parabola, be substituted for Props. III. and V.

Let D (fig. 14) be the pole with respect to a parabola of any straight line EF, and let the diameter through D meet EF in E, and the parabola in P; and let the intercepts on ED, EF of any conic harmonically circumscribed to the parabola be h, h'; k, k', then will

$$\frac{1}{h} + \frac{1}{h'} - \frac{2c}{hh'} + \frac{2a}{kk'} = 0,$$

where c and a are the distances of P from D and the focus respectively.

Because EF, ED and the polar of E from an infinitely long self-conjugate triangle, the lines MN, $M'N'$ joining

the points where the conic cuts ED, EF will be conjugate lines. Let $\alpha\beta$, the pole of MN, lie on $M'N'$, then taking ED, EF as axes

$y^2 = 4a(x-c)$ is the equation to the parabola,

$y\beta = 2a(x+\alpha-2c)$..... to MN.

Hence, $h+\alpha-2c=0$, $k\beta = 2a(\alpha-2c)$.

Also $$\frac{\alpha}{h'}+\frac{\beta}{k'}=1.$$

Eliminating α, β from these three equations the required condition is obtained.

17. The next proposition applies to a case in which the formula of Prop. III. fails.

If a conic S, harmonically circumscribed to another C, pass through the centre of C, the asymptotes of S will be parallel to conjugate diameters of C.

The self-conjugate triad of C whose vertex is at the centre of C has by Prop. I. its other angular points where the polar of the first point with respect to C cuts S. The line at infinity being this polar, the extremities of the asymptotes of S and the centre of C from a self-conjugate triad, whence the proposition follows.

Cor. 1. The centre of the circle inscribed in any self-conjugate triangle of a rectangular hyperbola lies on the hyperbola.

N.B. The above proposition is reciprocal to that in section 13.

18. If tangents from two points M, N to a conic C, touch it at P, Q, R, S, it is known that a conic F can be described through the six points M, N, P, Q, R, S. Let S be a conic passing through M, N and harmonically circumscribed to C, then if F and S intersect again in H, K, the poles of HK, MN with respect to S and F, will be poles of MN, HK with respect to F, S respectively.

The conic F, discussed in Salmon's *Conics*, Art. 383 (5th edition), becomes the director circle of C when M and N are the circular points at infinity. The proposition is a generalization from Prop. VI. Cor. 1.

19. From one of the properties of the conic F we deduce that: If the polar of any point O with respect to C cut S

in M and N, and OP be a tangent from O to S, the tangents from P to C together with the straight lines PM, PN form a harmonic pencil.

20. A simpler property and one more obviously connected with the director circle is: If the tangent at a point P on S pass through the centre of C, the tangents from P to C are parallel to conjugate diameters of S.

21. Problems relating to the point of intersection of the perpendiculars of a triangle inscribed in or circumscribed about a conic may sometimes be solved by considering that point as the centre of a circle with respect to which the triangle is self-conjugate. The circle is imaginary unless the triangle be obtuse the square of its radius being the product of the segments of the perpendiculars drawn *in the same direction* from its centre. The conics circumscribed about or inscribed in the triangle are harmonically circumscribed or inscribed to the circle.

Ex. (1). *The centre of perpendiculars of a triangle formed by three tangents to a parabola lies on the directrix.*

The parabola is harmonically inscribed in the above-mentioned circle, and therefore the circle harmonically circumscribed to the parabola. Its centre therefore lies on the directrix.

Ex. (2). *If the rectangle under the segments of one of the perpendiculars of the triangle formed by three tangents to a conic be M, the locus of the intersection of perpendiculars is the circle $x^2+y^2=a^2+b^2+M$.*

M is the square of the radius of the circle to which the conic is harmonically inscribed, and which cuts the director circle orthogonally, whence the proposition follows.

Ex. (3). *If the rectangle under the segments of one of the perpendiculars of a triangle inscribed in S be constant, and $= M$, the locus of intersection of perpendiculars is the conic concentric and similar to S, $S=M\left(\frac{1}{a^2}+\frac{1}{b^2}\right)$ (Dr. Hart).*

The conic is harmonically circumscribed to the circle with respect to which the triangle is self-conjugate.

Referring the conic to its axes, the condition that the circle should be harmonically inscribed in it is

$$\frac{x^2-r^2}{a^2}+\frac{y^2-r^2}{b^2}=1,$$

where $r^2 = M$, and x and y are the coordinates of the centre of the circle. The equation to the required locus is therefore

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = r^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right),$$

or

$$S = M\left(\frac{1}{a^2} + \frac{1}{b^2}\right),$$

whatever form the equation to the conic assumes by transformation.

Ex. (4). *Find the locus of the intersection of perpendiculars of a triangle inscribed in one conic and circumscribed about another.* (Mr. Burnside).

By combining the foregoing propositions and eliminating M, the equation is found to be

$$x^2 + y^2 - a^2 - b^2 = \frac{a'^2 b'^2}{a'^2 + b'^2} S.$$

See Salmon's *Conics*, Art. 375, (5th edition), from which these examples have been taken.

NOTE ON CONDUCTION OF HEAT.

By *W. L. Mollison, M.A.*

THE object of the present note, is to solve some problems in the variable motion of heat by the methods given by Professor Stokes in his paper "On the critical values of the sums of periodic series." The point of the method is that, given a function expanded in sines or cosines, its derivatives can be formed so as to allow for and to exhibit the discontinuity of the function. The articles in Stokes' paper which are directly used, are §§ 12, 28, 33.

1. Given a bar of finite length exposed to an atmosphere at temperature zero, the ends $x = 0$, $x = a$ being kept at temperatures $\phi(t)$, $\psi(t)$ respectively, and the initial temperature at distance x from an extremity being $F(x)$, it is required to find the temperature at any time t.

We know that if v be the temperature, and if $u = ve^{-kt}$

$$\frac{du}{dt} = \frac{d^2u}{dx^2}.$$

Let now $$u = \Sigma A_n \sin \frac{n\pi x}{a},$$

A_n being a function of t, then if

$$A_n = \frac{2}{a}\int_0^a f(x, t) \sin \frac{n\pi x}{a}\, dx,$$

the temperature at any point is $f(x, t)$. Now by integrating twice by parts

$$\int_0^a f(x, t) \sin \frac{n\pi x}{a}\, dx = \frac{a}{n\pi}\{f(0, t) - (-1)^n f(a, t)\} - \frac{a^2}{n^2\pi^2}\int_0^a \frac{d^2 f}{dx^2} \sin \frac{n\pi x}{a}\, dx.$$

Now by hypothesis $f(0, t)$, $f(a, t)$ (*i.e.* the temperatures at the extremities) are $\phi(t)$, and $\psi(t)$ respectively; therefore the coefficient of $\sin \frac{n\pi x}{a}$ in the expansion of $\frac{d^2 u}{dx^2}$ is

$$-\frac{n^2\pi^2}{a^2} A_n + \frac{2n\pi}{a^2}\{\phi(t) - (-1)^n \psi(t)\},$$

we therefore have to determine A_n from the equation

$$\frac{dA_n}{dt} + \frac{n^2\pi^2}{a^2} A_n = \frac{2n\pi}{a^2}\{\phi(t) - (-1)^n \psi(t)\};$$

whence we get

$$A_n = C_n e^{-\frac{n^2\pi^2}{a^2}t} + \frac{2n\pi}{a^2} e^{-\frac{n^2\pi^2}{a^2}t} \int_0^t e^{\frac{n^2\pi^2}{a^2}t'} \{\phi(t') - (-1)^n \psi(t')\}\, dt';$$

it only remains to determine C_n, and since when $t=0$, we have

$$u = \Sigma C_n \sin \frac{n\pi x}{a} = F(x),$$

it follows that

$$C_n = \frac{2}{a}\int_0^a F(x) \sin \frac{n\pi x}{a}\, dx;$$

therefore $$u = \sum_{n=1}^{n=\infty} C_n e^{-\frac{n^2\pi^2}{a^2}t} \sin \frac{n\pi x}{a}$$

$$+ \frac{2\pi}{a^2} \sum_{n=1}^{n=\infty} n e^{-\frac{n^2\pi^2}{a^2}t} \sin \frac{n\pi x}{a} \int_0^t e^{\frac{n^2\pi^2}{a^2}t'} \{\phi(t') - (-1)^n \psi(t')\}\, dt'.$$

2. Let the rod now extend indefinitely on the positive side of the origin. Let the temperature at $x=0$ be $\chi(t)$, and the initial temperature $F(x)$.

We know that

$$f(x)=\frac{2}{\pi}\int_0^\infty\int_0^\infty f(x')\sin\beta x\sin\beta x'\,d\beta\,dx'.$$

Let $$\phi(\beta)=\frac{2}{\pi}\int_0^\infty F(x')\sin\beta x'\,dx'.$$

Let now $$u=\int_0^\infty A_\beta\sin\beta x\,d\beta,$$

where $$A_\beta=\frac{2}{\pi}\int_0^\infty f(x',t)\sin\beta x'\,dx',$$

then the temperature at any point will be $f(x,t)$; now by integrating twice by parts, we get

$$\int_0^\infty f(x',t)\sin\beta x'dx'=\frac{1}{\beta}f(0,t)-\frac{1}{\beta^2}\int_0^\infty\frac{d^2f}{dx^2}\sin\beta x'dx',$$

therefore the coefficient of $\sin\beta x$ in the value of $\frac{d^2u}{dx^2}$ is

$$-\beta^2A_\beta+\frac{2}{\pi}\beta f(0,t),$$

but $f(0,t)=\chi(t)$ by hypothesis, and therefore the equation A_β must satisfy is

$$\frac{dA_\beta}{dt}+\beta^2A_\beta=\frac{2}{\pi}\beta\chi(t);$$

therefore $$A_\beta=C_\beta e^{-\beta^2t}+\frac{2}{\pi}\beta e^{-\beta^2t}\int_0^t e^{\beta^2t'}\chi(t')dt',$$

we see at once that $C_\beta=\phi(\beta)$, and therefore the temperature at any time t is

$$u=\int_0^\infty e^{-\beta^2t}\phi(\beta)\sin\beta x\,d\beta + \frac{2}{\pi}\int_0^\infty\int_0^t e^{-\beta^2(t-t')}\beta\chi(t')\sin\beta x\,d\beta\,dt'.$$

3. Let the sides $x=0$, $x=a$ of a rectangle be kept at temperatures $\phi_1(t,y)$, $\phi_2(t,y)$ respectively, and the sides $y=0$, $y=b$ at temperatures $\psi_1(t,x)$, $\psi_2(t,x)$ it is required to find the temperature at any point at any time.

The equation of conduction of heat is

$$\frac{dv}{dt} = \frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Let then

$$v = \Sigma\Sigma A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \quad \ldots\ldots\ldots\ldots\ldots(2),$$

m and n being integers; if then

$$A_{mn} = \frac{4}{ab}\int_0^a\int_0^b f(x, y, t) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dx\, dy,$$

the temperature at any point is $f(x, y, t)$.

Now, by integration by parts, we have

$$\int_0^a\int_0^b f(x, y, t) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dx\, dy$$

$$= \int_0^b -\frac{a}{m\pi}\{\cos m\pi f(a, y, t) - f(0, y, t)\} \sin\frac{n\pi y}{b}\, dy$$

$$-\frac{a^2}{m^2\pi^2}\int_0^a\int_0^b \frac{d^2f}{dx^2} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dx\, dy.$$

Now $$f(a, y, t) = \phi_2(t, y),$$

$$f(0, y, t) = \phi_1(t, y);$$

and, therefore,

$$\int_0^a\int_0^b f(x, y, t) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dx\, dy$$

$$= \frac{a}{m\pi}(-1)^m\int_0^b \phi_2(t, y) \sin\frac{n\pi y}{b}\, dy + \frac{a}{m\pi}\int_0^b \phi_1(t, y) \sin\frac{n\pi y}{b}\, dy$$

$$-\frac{a^2}{m^2\pi^2}\int_0^a\int_0^b \frac{d^2f}{dx^2} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dx\, dy,$$

denoting the coefficient of $\frac{m\pi}{a}$ in the first line of the right side by $\chi_1(t)$, we see that the coefficient of $\sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$ in $\frac{d^2v}{dx^2}$ is

$$-\frac{m^2\pi^2}{a^2} A_{mn} + \frac{4}{ab}\frac{m\pi}{a}\chi_1(t),$$

and so the coefficient of $\sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$ in $\frac{d^2v}{dy^2}$ is

$$= -\frac{n^2\pi^2}{b^2} A_{mn} + \frac{4}{ab}\frac{n\pi}{b}\chi_2(t),$$

where $\chi_2(t)$ is formed in a manner similar to $\chi_1(t)$ taking for the temperatures at $y=0$, $y=b$,

$$\psi_1(t, x), \quad \psi_2(t, x);$$

therefore the differential equation which A_{mn} must satisfy in order that (2) may be a solution of (1) under the given bounding conditions is

$$\frac{dA_{mn}}{dt} + \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) A_{mn} = \frac{4}{ab}\frac{m\pi}{a}\chi_1(t) + \frac{4}{ab}\frac{n\pi}{b}\chi_2(t)$$

$$= F(t) \text{ say};$$

therefore

$$A_{mn} = C_{mn} e^{-\pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)t} + e^{-\pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)t} \int_0^t e^{\pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)t'} F(t')\, dt',$$

if the initial heating is $v = f(x, y)$, we get

$$C_{mn} = \frac{4}{ab}\int_0^a \int_0^b f(x, y) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dx\, dy,$$

and $v = \Sigma\Sigma C_{mn} e^{-\pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)t} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$

$$+ \Sigma\Sigma e^{-\pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)t} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \int_0^t e^{\pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)t'} F(t')dt'.$$

If the temperatures, at which the sides are maintained, be functions of the time only, then the expressions $\chi_1(t)$, $\chi_2(t)$ can be at once written down in terms of $\phi_1(t)$, $\phi_2(t)$, $\psi_1(t)$, $\psi_2(t)$; in this case both m and n must be odd.

It is evident that exactly the same method will apply to the problem of a parallelopiped having its six faces kept at temperature which are arbitrary functions of the time and the coordinates of the points on the respective faces.

P.S. Since sending the above to the *Messenger* I find that the second problem has been solved by Professor Stokes in a letter to Sir W. Thomson, published in 1856 in the *Philosophical Magazine*, in relation to the variable motion of electricity.

ON A DIFFERENTIAL EQUATION.

By *H. J. Sharpe, M.A.*

HAVING had occasion to discuss the following differential equation in connexion with the theory of the reflection of sound, a certain point occurs in it which I thought would

have some interest for the readers of the *Messenger*.* The equation is

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}+y(x+A)=0 \quad \ldots\ldots\ldots\ldots (1)$$

where A is a constant. This equation can be solved in a series in the usual way. There are two solutions, each multiplied by an arbitary constant.

The first is a series of ascending powers of x. The second involves in addition $\log x$. In the following investigation we confine ourselves to the first solution

$$y=1-\frac{x^2}{2^2}+\frac{x^4}{2^2.4^2}-\&c.+\left\{-xA+\frac{x^3A^2}{4}-\frac{x^5}{36}(A^3-5A)\right.$$
$$\left.+\frac{x^4}{16\times 36}(A^4-14A^2)+\&c.\right\} \quad \ldots\ldots\ldots\ldots (2)$$

(1) can also be solved by definite integrals. The solution corresponding to (2) is

$$y=C\int_0^{\frac{1}{2}\pi}\cos(x\cos\theta+A\log\cot\tfrac{1}{2}\theta)\,d\theta \quad \ldots\ldots(3).$$

The other solution in a definite integral involves $\log x$. It must be possible to determine C so that (3) shall be identical with (2). To do this we observe that in (2) y reduces to unity when $x=0$. From (3) then we get

$$1=C\int_0^{\frac{1}{2}\pi}\cos(A\log\cot\tfrac{1}{2}\theta)\,d\theta \quad \ldots\ldots\ldots\ldots(4).$$

Put $\cot\frac{1}{2}\theta=\varepsilon^{\phi}$, then

$$\cos\theta=\frac{\varepsilon^{\phi}-\varepsilon^{-\phi}}{\varepsilon^{\phi}+\varepsilon^{-\phi}}$$

$$-\frac{d\theta}{d\phi}=\frac{2}{\varepsilon^{\phi}+\varepsilon^{-\phi}}$$

therefore (4) transforms into

$$1=2C\int_0^{\infty}\frac{\cos A\phi}{\varepsilon^{\phi}+\varepsilon^{-\phi}}\,d\phi \quad \ldots\ldots\ldots\ldots\ldots (5)$$

This definite integral has been evaluated in infinite terms. Its value can be at once deduced from p. 496 (ed. 1846) of

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* I am not aware, however, whether the results obtained are new or not.

Gregory's *Examples*, or from p. 669 of De Morgan's *Differential and Integral Calculus*. Then (5) becomes

$$1 = C\frac{\pi}{\varepsilon^{\frac{1}{2}(\pi A)} + \varepsilon^{-\frac{1}{2}(\pi A)}};$$

therefore (3) becomes

$$y = \frac{1}{\pi}\left\{\varepsilon^{\frac{1}{2}(\pi A)} + \varepsilon^{-\frac{1}{2}(\pi A)}\right\} \times \int_0^{\frac{1}{2}\pi} \cos\left(x\cos\theta + A\log\cot\tfrac{1}{2}\theta\right) d\theta \quad \ldots\ldots(6).$$

Thus the right-hand side of (2) is identical with the right-hand side of (6). Transforming (6) as we transformed (4), and putting for brevity

$$\varepsilon^{\phi} + \varepsilon^{-\phi} = s \quad \text{and} \quad \varepsilon^{\phi} - \varepsilon^{-\phi} = d$$

we get

$$y = \frac{2}{\pi}\left\{\varepsilon^{\frac{1}{2}(\pi A)} + \varepsilon^{-\frac{1}{2}(\pi A)}\right\} \times \int_0^{\infty} \cos\left(x\frac{d}{s} + A\phi\right)\frac{d\phi}{s} \quad \ldots\ldots(7).$$

If in (7) we expand the definite integral in ascending powers of A, this difficulty occurs that the series is ultimately divergent unless $A^2 < 1$, and then when this divergent series is multiplied by $\left\{\varepsilon^{\frac{1}{2}(\pi A)} + \varepsilon^{-\frac{1}{2}(\pi A)}\right\}$ the result expanded is of course still a divergent series. Thus we see that if (2) be arranged in ascending powers of A, the result is a divergent series unless $A^2 < 1$. I propose to discuss this difficulty and to show what the solution of (1) is when $A^2 > 1$. Let ϕ_1 be a value of ϕ, so large that for ϕ_1 and all larger values of ϕ, $\cos\left(x\frac{d}{s}\right)$ and $\sin\left(x\frac{d}{s}\right)$ do not change sign whatever x be.

Now in (7)

$$\int_0^{\infty} = \int_0^{\phi_1} + \int_{\phi_1}^{\infty} \quad \ldots\ldots(8).$$

The latter integral

$$= \int_{\phi_1}^{\infty}\left[\cos\left(x\frac{d}{s}\right) \times \left\{1 - \frac{1}{\lfloor 2}A^2\phi^2 + \&c.\right\}\right.$$

$$\left. - \sin\left(x\frac{d}{s}\right) \times \left\{A\phi - \frac{1}{\lfloor 3}A^3\phi^3 + \&c.\right\}\right]\frac{d\phi}{s} \quad \ldots\ldots(9).$$

When ϕ_1 is taken sufficiently large, the first part of this becomes approximately

$$= \cos x \int_{\phi_1}^{\infty}\left[1 - \frac{1}{\lfloor 2}A^2\phi^2 + \&c.\right]\frac{d\phi}{s},$$

and the general term of this integral is

$$\frac{(-1)^{\frac{1}{2}n}}{\lfloor n} A^n \int_{\phi_1}^{\infty} \phi^n \frac{d\phi}{s},$$

and $$\int_{\phi_1}^{\infty} \phi^n \frac{d\phi}{s} = \int_{\phi_1}^{\infty} \frac{\phi^n d\phi}{\varepsilon^{\phi}} - \int_{\phi_1}^{\infty} \frac{\phi^n \varepsilon^{-\phi} d\phi}{\varepsilon^{\phi}(\varepsilon^{\phi} + \varepsilon^{-\phi})} \quad \ldots\ldots (10).$$

The first of the integrals on the right-hand side $= \lfloor n$.

The second is numerically $< \int_{\phi_1}^{\infty} \phi^n \varepsilon^{-2\phi} d\phi$,

or $$< \frac{\lfloor n}{3^n} - \int_0^{\phi_1} \phi^n \varepsilon^{-2\phi} d\phi \quad \ldots\ldots\ldots\ldots\ldots (11).$$

The last integral is numerically $< \int_0^{\phi_1} \phi^n d\phi$ or $\frac{\phi_1^{n+1}}{n+1}$.

Since ϕ_1 is independent of n and finite, being in fact a function of x, which is supposed finite, we see that when n is large the first part of (11) is incomparably larger than the last, for when n is large, we have nearly

$$\lfloor n = (2\pi n)^{\frac{1}{2}} n^n \varepsilon^{-n}.$$

Thus $$\frac{\phi_1^{n+1}}{\lfloor n} \times \frac{3^n}{n+1} = \frac{\phi_1^{n+1}}{(2\pi n)^{\frac{1}{2}} n^n \varepsilon^{-n}} \times \frac{3^n}{n+1} \quad \ldots\ldots (12).$$

Now obviously ϕ_1 increases with x, but as x is supposed finite, ϕ_1 will be finite. Even if x be large, the right-hand side of (12) is less than what it becomes when ϕ_1 is put for ε, that is

$$< \frac{\phi_1}{(2\pi n)^{\frac{1}{2}}} \times \left(\frac{3\phi_1^2}{n}\right)^n \times \frac{1}{n+1},$$

which vanishes when n is very large.

We see thus that in (10) the first part of the right-hand side is incomparably largest when n is large, and thus when n is large the first part of (9) becomes approximately

$$(-1)^{\frac{1}{2}n} \cos x \{A^n - A^{n+2} + \&c.\},$$

n being even. Similarly the second part becomes

$$(-1)^{\frac{1}{2}(n+1)} \sin x \{A^n - A^{n+2} + \&c.\},$$

n being odd. We see thus that the integral in (7) when expanded in ascending powers of A is divergent unless $A^2 < 1$,

and therefore when $A^2 > 1$ is unreliable for purposes of numerical calculation. The difficulty may be paralleled thus. We know that if $\int_0^\infty \varepsilon^{-\phi} \cos A\phi \, d\phi$ be expanded in ascending powers of A, the series is divergent unless $A^2 < 1$ for the integral

$$= \frac{1}{1+A^2}.$$

This suggested the idea whether the integral in (7) could be expressed in the form

$$\frac{f(A)}{1+A^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (13)$$

where $f(A)$ is a convergent series. On trial however, I found $f(A)$ to be still divergent if $A^2 > 1$.

Call the integral in (7) X. I shall now prove that

$$X = \frac{f(A)}{\varepsilon^{\frac{1}{2}(\pi A)} + \varepsilon^{-\frac{1}{2}(\pi A)}} \quad \ldots\ldots\ldots\ldots\ldots \quad (14)$$

where $f(A)$ is a *convergent* series for all values of x and A. We may observe that the denominator in (14)

$$= 2(1+A^2)\left(1+\frac{A^2}{3^2}\right)\left(1+\frac{A^2}{5^2}\right) \ldots\ldots$$

We see thus that in the denominator of (14) there are an infinite number of factors analogous to the denominator of (13). The required transformation is effected by continually integrating by parts the left-hand side of the equation

$$\int_0^\infty \cos A\phi \frac{d\phi}{s} = \frac{\frac{1}{2}\pi}{\varepsilon^{\frac{1}{2}(\pi A)} + \varepsilon^{-\frac{1}{2}(\pi A)}}.$$

Put for shortness $\frac{d}{d\phi} = \Delta$, then

$$\frac{\frac{1}{2}\pi}{\varepsilon^{\frac{1}{2}(\pi A)} + \varepsilon^{-\frac{1}{2}(\pi A)}} = -\frac{1}{A}\int_0^\infty \sin A\phi \, \Delta \frac{1}{s} d\phi$$

$$= -\frac{1}{A^2}\int_0^\infty \cos A\phi \, \Delta^2 \frac{1}{s} d\phi = \&c.$$

$$= \frac{1}{A^n}\int^\infty \cos\left(A\phi + \frac{n\pi}{2}\right) \Delta^n \frac{1}{s} d\phi \ldots\ldots (15).$$

Put the integral in (7) $= X$, then

$$X = \int_0^\infty \left(\frac{1}{s} - \frac{1}{\lfloor 2}x^2\frac{d^2}{s^3} + \frac{1}{\lfloor 4}x^4\frac{d^4}{s^5} - \&c.\right)\cos A\phi d\phi$$

$$- \int_0^\infty \left(x\frac{d}{s^2} - \frac{x^3}{\lfloor 3}\frac{d^3}{s^4} + \frac{x^5}{\lfloor 5}\frac{d^5}{s^6} - \&c.\right)\sin A\phi d\phi \ldots (16)$$

Put for shortness $\frac{d^m}{s^{m+1}} = \delta_m$ then we can readily show that

$$\frac{d}{d\phi}\delta_m = m\delta_{m-1} - (m+1)\delta_{m+1} \ldots\ldots\ldots (17)$$

In (17) by putting m successively $= 0, 1, 2, 3$ &c. we can readily establish the following

$$\left.\begin{aligned} \delta_1 &= -\Delta\frac{1}{s}, \quad \delta_2 = \frac{1+\Delta^2}{2}\frac{1}{s}, \quad \delta_3 = -\frac{5\Delta+\Delta^3}{6}\frac{1}{s} \\ \delta_4 &= \frac{9+14\Delta^2+\Delta^4}{24}\frac{1}{s}, \quad \delta_5 = -\frac{89\Delta+30\Delta^3+\Delta^5}{120}\frac{1}{s} \end{aligned}\right\} \ldots (18)$$

$$\left.\delta_6 = \frac{225+439\Delta^2+55\Delta^4+\Delta^6}{720}\right\} \ldots\ldots\ldots (18)$$

This shows that we may assume generally

$$\delta_m = (-1)^m\left[(m,m)\Delta^m + (m,m-2)\Delta^{m-2} + (m,m-4)\Delta^{m-4} + \&c.\right]\frac{1}{s} \quad (19)$$

where (m, m) &c. are functions of m to be determined.

If m is even, the last term in brackets is $(m, 0)$.

If m is odd, the last term in brackets is $(m, 1)\Delta$.

Substitute from (19) into (17) and equating the coefficients of corresponding powers of Δ we get the following functional equations for determingin (m, m) &c.

$$\left.\begin{aligned} (m, m) &= (m+1)(m+1, m+1) \\ (m, m-2) &= -m(m-1, m-1) + (m+1)(m+1, m-1) \\ (m, m-4) &= -m(m-1, m-3) + (m+1)(m+1, m-3) \\ \&c. &= \&c. \end{aligned}\right\} \ldots (20)$$

If m be even, the last equations will be

$$(m, 2) = -m(m-1, 3) + (m+1)(m+1, 3)$$

$$(m, 0) = -m(m-1, 1) + (m+1)(m+1, 1).$$

If m be odd the last equations will be

$$(m, 1) = -m(m-1, 2) + (m+1)(m+1, 2),$$
$$0 = -m(m-1, 0) + (m+1)(m+1, 0).$$

These equations can be readily solved from either end, but it will be found simpler to solve them from the beginning. It will be found that

$$\left.\begin{aligned} (m, m) &= \frac{1}{\lfloor \underline{m} \quad}, (m, m-2) = \frac{1}{3\lfloor \underline{m-3}} + \frac{1}{2\lfloor \underline{m-2}} \\ (m, m-4) &= \frac{1}{18\lfloor \underline{m-6}} + \frac{11}{30\lfloor \underline{m-5}} + \frac{3}{8\lfloor \underline{m-4}} \\ &\text{\&c.} \qquad\qquad \text{\&c.} \end{aligned}\right\} \ldots(21)$$

Substituting from (18) and (19) into (16) we get

$$X = \int_0^\infty \left[1 - \frac{x^2}{\lfloor \underline{2}} \tfrac{1}{2}(\Delta^2 + 1) + \frac{x^4}{\lfloor \underline{4}} \tfrac{1}{24}(\Delta^4 + 14\Delta^2 + 9) - \text{\&c.}\right.$$
$$\left. + (-1)^{\frac{1}{2}m} \frac{x^m}{\lfloor \underline{m}} \{(m, m)\Delta^m + (m, m-2)\Delta^{m-2} + \text{\&c.} + (m, 0)\} + \text{\&c.}\right] \times \frac{1}{s} \cos A\phi d\phi$$

$$+ \int_0^\infty \left[x\Delta - \frac{x^3}{\lfloor \underline{3}} \tfrac{1}{6}(\Delta^3 + 5\Delta) + \frac{x^5}{\lfloor \underline{5}} \tfrac{1}{120}(\Delta^5 + 30\Delta^3 + 89\Delta) - \text{\&c.}\right.$$
$$\left. + (-1)^{\frac{1}{2}(m-1)} \frac{x^m}{\lfloor \underline{m}} \{(m, m)\Delta^m + (m, m-2)\Delta^{m-2} + \text{\&c.} + (m, 1)\Delta\} + \text{\&c.}\right] \times \frac{1}{s} \sin A\phi d\phi.$$

Substitute from equations (15) into this, then

$$X\frac{2}{\pi}(\varepsilon^{\frac{1}{2}\pi A} + \varepsilon^{-\frac{1}{2}\pi A}) = y$$
$$= 1 - \frac{x^2}{(\lfloor \underline{2})^2}(-A^2 + 1) + \frac{x^4}{(\lfloor \underline{4})^2}(A^4 - 14A^2 + 9) - \text{\&c.}$$
$$+ \frac{x^m}{\lfloor \underline{m}} \{(m, m)A^m - (m, m-2)A^{m-2} + \text{\&c.} + (-1)^{\frac{1}{2}m}(m, 0)\} + \text{\&c.}$$
$$- xA - \frac{x^3}{(\lfloor \underline{3})^2}(A^3 - 5A) + \frac{x^5}{(\lfloor \underline{5})^2}(-A^5 + 30A^3 - 89A) - \text{\&c.}$$
$$- \frac{x^m}{\lfloor \underline{m}} \{(m, m)A^m - (m, m-2)A^{m-2} + \text{\&c.} + (-1)^{\frac{1}{2}(m-1)}(m, 1)A\}$$
$$+ \text{\&c.} \ldots(22).$$

This is the required transformation of X. I will now shew that the right-hand side is convergent for all finite values of A and x. It is evident that (16) is convergent, provided that we do not expand the right-hand side in powers of A. But if we divide each side of (22) by

$$\frac{2}{\pi}\left(\varepsilon^{\frac{1}{2}\pi A}+\varepsilon^{-\frac{1}{2}\pi A}\right),$$

(22) will be identical with (16). A more formal proof might be given, but there is no need. We may notice in passing in each of equations (18) that the sum of the coefficients on the right-hand side is alternately -1 and $+1$. It is easy to shew from equations (20) that this property is generally true. Hence all quantities of the type of (m, r) are < 1.

(22) may be arranged in a form very suitable for numerical calculation, especially when A is large. The method consists in arranging together all the 1st terms in the brackets, and summing them, then all the 2nd terms and all the 3rd, and so on till the series is exhausted. Call these results X_1, X_2, X_3, &c., and put $Ax = z$, then

$$X_1 = 1 - z + \frac{z^2}{(\lfloor 2)^2} - \frac{z^3}{(\lfloor 3)^2} + \&c. = J_0(2\sqrt{z}) \ldots\ldots (23),$$

or

$$= \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \cos(2\sqrt{z}\cos\theta)\, d\theta,$$

X_2 is a series beginning with $-\frac{x^2}{(\lfloor 2)^2}$ and of which the general term is

$$(-1)^{m+1}\frac{x^2 z^{m-2}}{\lfloor m}(m, m-2).$$

X_2 can be readily expressed in terms of differential coefficients X_1 with regard to z. By (21)

$$(-1)^{m+1}\frac{z^{m-2}}{\lfloor m}(m, m-2) = (-1)^{m+1}\frac{z^{m-2}}{\lfloor m}\left\{\frac{1}{3\lfloor m-3} + \frac{1}{2\lfloor m-2}\right\}.$$

In (23) put $\qquad u_m = (-1)^m \frac{z^m}{(\lfloor m)^2}.$

Then

$$\frac{d^2 u_m}{dz^2} = (-1)^m \frac{z^{m-2}}{\lfloor m \lfloor m-3}, \quad \frac{d^2 u_m}{dz^2} = (-1)^m \frac{z^{m-2}}{\lfloor m \lfloor m-2}.$$

therefore

$$-\tfrac{1}{8}z\frac{d^3u_m}{dz^3}-\tfrac{1}{2}\frac{d^2u_m}{dz^2}=(-1)^{m+1}\frac{z^{m-4}}{\lfloor m}(m,\,m-2).$$

Sum this from $m=2$ to $m=\infty$, and we get

$$X_2=-\tfrac{1}{2}\frac{d^2X_1}{dz^2}x^2-\tfrac{1}{8}\frac{d^3X_1}{dz^3}x^2z \ldots\ldots\ldots (24).$$

In (22) X_2 is a series beginning with $\frac{9x^4}{(\lfloor 4)^2}$ and of which the general term is $(-1)^m\frac{x^4z^{m-4}}{\lfloor m}(m,\,m-4)$. By (21)

$$(-1)^m\frac{z^{m-4}}{\lfloor m}(m,\,m-4)$$

$$=(-1)^m\frac{z^{m-4}}{\lfloor m}\left\{\frac{1}{18\lfloor m-6}+\frac{11}{30\lfloor m-5}+\frac{3}{8\lfloor m-4}\right\}.$$

In (23) putting as before

$$u_m=(-1)^m\frac{z^m}{(\lfloor m)^2},\quad \frac{d^2u_m}{dz^2}=(-1)^m\frac{z^{m-2}}{\lfloor m\lfloor m-6},$$

$$\frac{d^3u_m}{dz^3}=(-1)^m\frac{z^{m-3}}{\lfloor m\lfloor m-5},\quad \frac{d^4u_m}{dz^4}=(-1)^m\frac{z^{m-4}}{\lfloor m\lfloor m-4},$$

therefore

$$\frac{z^2}{18}\frac{d^6u_m}{dz^6}+\frac{11z}{30}\frac{d^5u_m}{dz^5}+\frac{3}{8}\frac{d^4u_m}{dz^4}=(-1)^m\frac{z^{m-4}}{\lfloor m}(m,\,m-4).$$

Sum this from $m=4$ to $m=\infty$ and we get

$$X_2=\left(\frac{3}{8}\frac{d^4X_1}{dz^4}+\frac{11z}{30}\frac{d^5X_1}{dz^5}+\frac{z^2}{18}\frac{d^6X_1}{dz^6}\right)x^4 \ldots (25),$$

and so we can express X_4, &c. in terms of differential coefficients of X_1. The transformations (23), (24), (25), &c. of (22) give us series for the calculation of X for any given value of x, and for values of A such that $A^2>1$. They are very rapidly convergent for large values of A and small values of x. Supposing z very large in comparison with x

we may use the transformation in Art. 408 of Todhunter's *Laplace's Functions*. Then from (23)

$$\pi X_1 = \frac{\sqrt{(2\pi)}}{\sqrt{(2)}z^{\frac{1}{2}}} \cos(2z^{\frac{1}{2}} - \tfrac{1}{4}\pi) \text{ nearly.}$$

If we substitute this in the series X_2, X_4, &c., we shall find the terms continually diminish, so that if we equate to 0 the right-hand side of (22) and consider it as an equation in A, whatever value x have, the large roots of the equations are approximately given by the formula

$$2\sqrt{(Ax)} - \tfrac{1}{4}\pi = (2n+1)\tfrac{1}{2}\pi.$$

We may also remark that if we have the equation

$$\mu X + \frac{dX}{dx} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (26),$$

where X has value in (22) and μ is any arbitrary constant, which may be 0 or ∞, whether we consider (26) as an equation in A, x being given; or as an equation in x, A being given, in either case the roots are all real, analogous to the property of the equation $J_0(x) = 0$. We will give only the proof of the first case. If X_1, X_2 are 2 values of X corresponding to 2 different roots A_1, A_2 of (26), x being given, it is easy to shew from (1) that

$$x\left(X_2\frac{dX_1}{dx} - X_1\frac{dX_2}{dx}\right) + (A_2 - A_1)\int X_1, X_2\, dx = 0.$$

If the integral be taken from $x = 0$ up to the particular value of x given in (26) the integrated part vanishes if we have

$$(A_2 - A_1)\int_0^x X_1, X_2\, dx = 0 \ldots\ldots\ldots\ldots (27).$$

If A_1, A_2 are a pair of conjugate impossible roots (27) cannot be satisfied.

We may further remark that however large x may be, the series in (22), when arranged in the form

$$X_1 + X_2 + X_3 + \text{&c.},$$

may be altered into another form still more convenient for numerical calculation. For putting for shortness $\frac{d}{dz} = d$ the series is

$$\left[1 - (\tfrac{1}{2}d^2 + \tfrac{1}{3}zd^3)x^2 + \left(\tfrac{3}{8}d^4 + \tfrac{11}{8}zd^5 + \frac{z^2}{18}d^6\right)x^4 - \text{&c.}\right] X_1.$$

Now we know that the series $a_1x^2 - a_2x^4 + a_3x^6 - \&c.$ if convergent and when the terms are alternately positive and negative may be transformed into the still more rapidly converging series

$$\frac{a_1x^2}{1+x^2} - \Delta a_1 \frac{x^4}{(1+x^2)^2} + \Delta^2 a_1 \frac{x^6}{(1+x^2)^3} - \&c.,$$

where

$$\Delta a_m = a_{m+1} - a_m.$$

When the first part of this article was in print and it was too late to alter it, I noticed that there is an error in it, which fortunately can be easily remedied and which does not affect the main result, but, at the same time, an error which must be corrected. On p. 176, I said "Thus we see that if (2) be arranged in ascending powers of A, the result is a divergent series unless $A^2 < 1$." This is untrue and should be omitted. (2) is convergent for all values of x and A, however large, as we may readily see thus: If we call the right-hand side of (2)

$$a_0 + a_1x + a_2x^2 + \&c. \quad \ldots\ldots\ldots\ldots\ldots\ldots (28),$$

we can readily show that

$$(n+1)^2 a_{n+1} + Aa_n + a_{n-1} = 0$$

which shows that (2) is ultimately convergent for *all* values of x and A, although for large values of A it begins by diverging. Again, we see this by comparing together equations (2) and (22). Still further it can be shown that (22) can be deduced directly from (2) without the elaborate transformation of the definite integral given on pp. 178–180. I will briefly indicate the process. In (28) a_0, a_1, &c. are all functions of A. If we actually examine the coefficients of the first few powers of x, up say to x^6, we shall find that generally when n is even a_n is of the form

$$(n, n)A^n - (n, n-2)A^{n-2} + \&c. + (-1)^{\frac{1}{2}n}\ (n, 0),$$

and when n is odd a_n is of the form

$$-(n, n)A^n + (n, n-2)A^{n-2} - \&c. + (-1)^{\frac{1}{2}(n+1)}(n, 1)A,$$

where (n, n), &c. are functions of n to be determined. Substituting then the corresponding values of a_{n+1}, a, a_{n-1} in (28) and equating to 0 the successive powers of A,

beginning with the highest, we shall find, whether n be odd or even, the following equations to determine (n, n), &c.

$$(n+1)^2(n+1, n+1) - (n, n) = 0,$$
$$(n+1)^2(n+1, n-1) - (n, n-2) - (n-1, n-1) = 0,$$
$$(n+1)^2(n+1, n-3) - (n, n-4) - (n-1, n-3) = 0,$$
$$\&c. \qquad \&c.$$

When these equations are solved, which they can easily be, and the values of (n, n), &c. substituted in (28), it will be found that (28) is identical with (22). This will be found to be a much shorter way of arriving at (22). But still perhaps the investigations on pp. 178–180 may have some value, as shewing how a difficulty was overcome.

THE METHOD OF LEAST SQUARES APPLIED TO CONDITIONED OBSERVATIONS.

By *A. E. Steinthal, B.A.*

THE method of least squares is originally due to Gauss, who was the first to place it on a scientific basis, though the idea had been used by Legendre so early as 1805. His statement of the subject is spread over several papers—*Theoria Motus Corporum Coelestium*, *Disquisitio de Orbitu Palladis*, *Theoria Combinationis Observationum*, and a supplement to this last. The theory was first put into a systematic form by Encke in the *Berliner Astron. Jahrbuch* for 1834-5-6, following whom Chauvenet has, in an Appendix to his Astronomy, given a clear statement of the theory.

The account there given of the method of treatment of conditioned equations is a complete practical solution of the problem as far as concerns finding the values of the unknowns, but it does not contain any rule for finding the probable errors of the results. This is given without much proof in Merriman's "*Method of Least Squares*"; in what follows I have extended the proof given by Gauss for the case of unconditioned observations to the case where there are certain exact equations to be satisfied. It will be seen that in method and notation I have adhered very closely to the original.

Suppose we have a series of observations of given weight and certain exact equations which the variables must satisfy, it is required to find the most probable values of the variables, and the probable error.

Suppose we have m equations of condition

$$\left.\begin{aligned} a_1x + b_1y + \ldots - n_1 &= 0 \\ a_2x + b_2y + \ldots - n_2 &= 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots & \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (\mathrm{I}),$$

and s exact equations

$$\left.\begin{aligned} A_1x + B_1y + \ldots - N_1 &= 0 \\ A_2x + B_2y + \ldots - N_2 &= 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots & \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (\mathrm{II}),$$

the number of unknowns being μ. Let the mean error of n_1 be ε_1, of n_2, ε_2 and so on, then the weights being p_1, p_2 &c., we have, if ε denote the mean error of a standard observation

$$\varepsilon^2 = p_1\varepsilon_1^2 = p_2\varepsilon_2^2 = \ldots$$

To get the most probable values of $x, y\ldots$ we have to make

$$p_1(a_1x + b_1y + \ldots - n_1)^2 + p_2(a_2x + b_2y + \ldots - n_2)^2 + \ldots$$

a minimum subject to the s conditions (II).

Using indeterminate multipliers $\lambda_1, \lambda_2, \ldots$, and calling these most probable values $x_0, y_0, \ldots$, we have the equations

$$\left.\begin{aligned} [paa]x_0 + [pba]y_0 + \ldots - [pan] + A_1\lambda_1 + A_2\lambda_2 + \ldots &= 0 \\ [pab]x_0 + [pbb]y_0 + \ldots - [pbn] + B_1\lambda_1 + B_2\lambda_2 + \ldots &= 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots & \\ A_1x_0 + B_1y_0 + \ldots - N_1 &= 0 \\ A_2x_0 + B_2y_0 + \ldots - N_2 &= 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots & \end{aligned}\right\} (\mathrm{III}),$$

in all $\mu + s$ equations, which are just sufficient to determine the μ unknowns and the s additional letters denoted by λ's.

To solve these, multiply the first equation by Q_1, the second by Q_2, &c. and add. Then

$$\left.\begin{aligned} &\{[paa]Q_1 + [pba]Q_2 + \ldots + A_1Q_{\mu+1} + A_2Q_{\mu+2} + \ldots\}x \\ &+ \{[pba]Q_1 + [pbb]Q_2 + \ldots + B_1Q_{\mu+1} + B_2Q_{\mu+2} + \ldots\}y \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots \\ &+ \{A_1Q_1 + B_1Q_2 + \ldots\}\lambda_1 + \{A_2Q_1 + B_2Q_2 + \ldots\}\lambda_2 + \ldots \end{aligned}\right\} = \left\{\begin{aligned} &[pan]Q_1 \\ &+ [pbn]Q_2 \\ &+ \ldots\ldots\ldots \\ &+ Q_{\mu+1}N_1 \\ &+ Q_{\mu+2}F_2 \\ &+ \ldots\ldots\ldots \end{aligned}\right.$$

The $(\mu + s)$ Q's being at our disposal, let us choose then to make the coefficient of x unity and those of y, $z\ldots$ and the λ's to vanish. We get for the Q's a system of equations differing from (III) only by having 1 for $[pan]$ and 0 for $[pbn]\ldots$ and $N_1N_2\ldots$

With these values of the multipliers we have for the most probable value of x

$$x_0 = \xi_0 + \alpha_1 n_1 + \alpha_2 n_2 + \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where

$$\left.\begin{aligned} \xi_0 &= Q_{\mu+1} N_1 + Q_{\mu+2} N_2 + \ldots \\ \alpha_1 &= p_1 a_1 Q_1 + p_1 b_1 Q_2 + \ldots \\ \alpha_2 &= p_2 a_2 Q_1 + p_2 b_2 Q_2 + \ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\}$$

Now if ε_x denote the mean error of x_0, we have

$$\varepsilon_x^2 = \alpha_1^2 \varepsilon_1^2 + \alpha_2^2 \varepsilon_2^2 + \ldots$$

$$= \left\{\frac{\alpha_1^2}{p_1} + \frac{\alpha_2^2}{p_2} + \ldots\right\} \varepsilon^2 = \left[\frac{\alpha\alpha}{p}\right] \varepsilon^2.$$

But from the above equations we have

$$[a\alpha] = [paa]\, Q_1 + [pab]\, Q_2 + \ldots$$
$$= 1 - A_1 Q_{\mu+1} - A_2 Q_{\mu+2} - \ldots$$
$$[b\alpha] = [pba]\, Q_1 + [pbb]\, Q_2 + \ldots$$
$$= -B_1 Q_{\mu+1} - B_2 Q_{\mu+2} - \ldots$$

and therefore

$$\left[\frac{\alpha\alpha}{p}\right] = [a\alpha]\, Q_1 + [b\alpha]\, Q_2 + \ldots$$
$$= Q_1 - Q_{\mu+1} [A_1 Q_1 + A_2 Q_2 + \ldots] - \ldots$$
$$= Q_1$$

whence $\varepsilon_x^2 = Q_1 \varepsilon^2$.

This gives us a rule for finding the weight which is almost identical with that when there are no exact equations, and which, of course, includes that as a particular case.

Our next step is to calculate the value of ε.

Suppose the true values of the variables are $x_0 + \xi$, $y_0 + \eta$, ... and let us write

$$\left.\begin{aligned} a_1 x_0 + b_1 y_0 + \ldots - n_1 &= v_1 \\ a_2 x_0 + b_2 y_0 + \ldots - n_2 &= v_2 \\ &\ldots\ldots\ldots\ldots \end{aligned}\right\} \ldots\ldots\ldots\ldots (A),$$

and

$$\left.\begin{aligned} a_1 (x_0 + \xi) + b_1 (y_0 + \eta) + \ldots - n_1 &= u_1 \\ a_2 (x_0 + \xi) + b_2 (y_0 + \eta) + \ldots - n_2 &= u_2 \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \ldots\ldots\ldots (B),$$

so that $v_1, v_2, \ldots$ are the residual, and $u_1, u_2, \ldots$ the actual errors.

Since $x_0, y_0, \ldots$ and $x_0+\xi, y_0+\eta, \ldots$ both satisfy the system of exact equations, we have

$$\left.\begin{aligned} A_1\xi + B_1\eta + \ldots = 0 \\ A_2\xi + B_2\eta + \ldots = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (C).$$

Multiply equations (B) by p_1a_1, p_2a_2, &c. and add, then by p_1b_1, p_2b_2, &c. and add; we have, by means of equations (III),

$$\left.\begin{aligned} [paa]\xi + [pab]\eta + \ldots - [pau] - A_1\lambda_1 - A_2\lambda_2 - \ldots = 0 \\ [pba]\xi + [pbb]\eta + \ldots - [pbu] - B_1\lambda_1 - B_2\lambda_2 - \ldots = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\},$$

so that by comparison with (III) and (1)

$$\xi = \alpha_1 u_1 + \alpha_2 u_2 + \ldots,$$

where $\alpha_1, \alpha_2, \ldots$ denote the same as before.

Now from equations (A), we have

$$\left.\begin{aligned} [pav] = -A_1\lambda_1 - A_2\lambda_2 - \ldots \\ [pbv] = -B_1\lambda_1 - B_2\lambda_2 - \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\},$$

and from equations (B), we have by means of (III),

$$[pvv] + [pav]\xi + [pbv]\eta + \ldots = [puv],$$

i.e.

$$\left.\begin{aligned} [puv] = [pvv] - \lambda_1[A_1\xi + B_1\eta + \ldots] \\ - \lambda_2[A_2\xi + B_2\eta + \ldots] \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} = [pvv].$$

Also from equations (B)

$$\begin{aligned} [puu] &= [puv] + [pau]\xi + [pbu]\eta + \ldots \\ &= [pvv] + [pau]\xi + [pbu]\eta + \ldots. \end{aligned}$$

Now

$$\left.\begin{aligned} [pau] &= p_1a_1u_1 + p_2a_2u_2 + \ldots \\ \xi &= \alpha_1u_1 + \alpha_2u_2 + \ldots \end{aligned}\right\}.$$

In taking the product we may neglect any term which contains the product of two different u's, as, having no clue to the sign, the most probable value is zero. And, as a good approximation, we may take the actual errors of $x_0, y_0, \ldots$ as equal to the mean errors, so that on the whole we have

$$\begin{aligned} \xi[pau] &= p_1a_1\alpha_1\varepsilon_1^2 + p_2a_2\alpha_2\varepsilon_2^2 + \ldots \\ &= a_1\alpha_1\varepsilon^2 + a_2\alpha_2\varepsilon^2 + \ldots \\ &= [a\alpha]\varepsilon^2 \\ &= [1 - A_1Q_{\mu+1} - A_2Q_{\mu+2} - \ldots]\varepsilon^2. \end{aligned}$$

Suppose that the multipliers which we should have used in finding the value of y_0 are denoted by letters R, then we should have similarly

$$\eta\,[pbu] = [1 - B_1R_{\mu+1} - B_2R_{\mu+2} - \dots]\,\varepsilon^2,$$

and so on.

Thus, since $[puu] = m\varepsilon^2$, we should have

$$\begin{aligned} m\varepsilon^2 = [pvv] + \mu\varepsilon^2 - \varepsilon^2 [&A_1Q_{\mu+1} + A_2Q_{\mu+2} + \dots \\ &+ B_1R_{\mu+1} + B_2R_{\mu+2} + \dots \\ &+ \dots\dots\dots\dots\dots\dots\dots]. \end{aligned}$$

Consider the equations

$$\begin{aligned} &[paa]\,P_1 + [pba]\,P + \dots + A_1P_{\mu+1} + A_2P_{\mu+2} + \dots = \sigma_1, \\ &[pba]\,P_1 + [pbb]\,P_2 + \dots + B_1P_{\mu+1} + B_2P_{\mu+2} + \dots = \sigma_2, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &A_1P_1 + A_2P_2 + \dots = 0, \\ &B_1P_1 + B_2P_2 + \dots = 0, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}$$

Then, when $\sigma_1 = 1$, $\sigma_2 = 0$, &c., the P's are what we have denoted by Q's; if $\sigma_1 = 0$, $\sigma_2 = 1$, &c., then they are our R's and so on.

Now using multipliers and equating to zero the coefficients of all except $P_{\mu+1}$ the coefficient of which is put equal to unity, we have

$$P_{\mu+1} = \sigma_1I_1 + \sigma_2I_2 + \dots,$$

where
$$\begin{aligned} &[paa]\,I_1 + [pba]\,I_2 + \dots + A_1I_{\mu+1} + A_2I_{\mu+2} + \dots = 0, \\ &[pba]\,I_1 + [pbb]\,I_2 + \dots + B_1I_{\mu+1} + B_2I_{\mu+2} + \dots = 0, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &A_1I_1 + B_1I_2 + \dots = 1, \\ &A_2I_1 + B_2I_2 + \dots = 0. \end{aligned}$$

Thus we see that
$$\left.\begin{aligned} Q_{\mu+1} &= I_1 \\ R_{\mu+1} &= I_2 \\ &\dots\dots \end{aligned}\right\},$$

and that
$$A_1Q_{\mu+1} + B_1R_{\mu+1} + \dots = 1.$$

So too
$$A_2Q_{\mu+2} + B_2R_{\mu+2} + \dots = 1,$$

and so on; and finally we have

$$m\varepsilon^2 = [pvv] + \mu\varepsilon^2 - s\varepsilon^2$$

$$\varepsilon^2 = \frac{[pvv]}{m - \mu + s}$$

$$\varepsilon = \sqrt{\left(\frac{[pvv]}{m - \mu + s}\right)},$$

the expression for the mean error in terms of the residual errors.

The exact equations as well as the equations of condition have been taken in a linear form. If they are not given so, we must, before applying the method, obtain approximate values by some other means, and then by application of Taylor's theorem (taking the corrections to our first approximations as the unknowns) we shall have a system of linear equations as assumed in the previous work.

MATHEMATICAL NOTES.

A property of pedal curves.

In the Senate-House Examination, 1881, the following problem was proposed:

"To every point of a circular ring an inextensible string is attached just long enough to reach a fixed point outside the circle where all the strings are gathered into a knot. If the knot be now moved about in the plane of the circle so that some of the strings are always tense, prove that it traces a limaçon."

If S be the external point and T any point on the perimeter of the circle the locus of S will clearly be the envelope of circles, with centre T and radius TS. The result is found to be a limaçon, and the only strings which affect the problem are those between the tangents from S giving the inner loop. The following appears to be a complete solution of the problem.

Draw SQ perpendicular to the tangent at T and produce it to P so that $QP = QS$; then P is a point on the locus, for if T_1 and T_{11} be points taken on the circle on either side of T, PT_1, PT_{11} are clearly less than ST_1, ST_{11} and $PT = ST$, and therefore every string will be loose when S is moved to P except ST or PT.

But the locus of Q is a limaçon, being the pedal of the given circle, and therefore the locus of P is the double vector limaçon; the conditions of the problem only give the inner loop. But if these strings be removed and every

point on the concave portion of the perimeter be joined to S by an elastic string, we see at once the mathematical meaning of the outer loop, for, reasoning from a similar construction, we see that now PT is the only string which is unstretched, and that if S be moved so that every string except one is stretched, it will lie on the outer loop of the limaçon.

Treated thus as a question of pedals, this admits of easy generalisation; for consider any plane curve and any point in its plane, and every point of the former to be joined to the latter by strings, elastic or inelastic, according as the curve is concave or convex towards the point; then the point moved as above, separately for each concave and convex portion, will describe the double vector pedal of the curve, with respect to the point, of as many loops as tangents can be drawn from the point.

Thus if the point be inside an ellipse, the strings are all elastic and their junction must be moved quite outside the curve before a point on the double vector pedal is reached.

And we may conceive any curve to be thus generated by means of strings radiating from any point to every point of the first negative pedal of the half vector curve, the point taken being the pedal and vector origin.

The double pedal corresponding to the convex portion of a curve can be practically described by means of a piece of cloth attached to the curve so as exactly to fill the space between the tangents from the given point and the curve, a pencil being fixed at the apex; the cloth limits the movement of the pencil in a manner similar to that of inelastic strings.

P. A. MacMahon.

Maxima and Minima.

It is well known that we get maxima and minima values of a function for infinite values of its derived function, but it is not mentioned in any of the text books that in this case the mark which distinguishes a maximum from a minimum is not the same as in the ordinary case.

A maximum value of a function occurs when its derived function changes from positive to negative. If this takes place at a passage through zero the derived function is of course diminishing, and the second derived function is therefore negative.

But if it takes place at an infinite value the derived function is increasing, and, therefore the second derived function is positive.

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No. CIX.] NEW SERIES. [May, 1880.

THE

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W. ALLEN WHITWORTH, M.A.,

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Of course the second derived function will also be infinite, but in a case where we have a real maximum or minimum it will be possible to determine the sign; where there is a doubt we have a case parallel to the vanishing of the second derived function in the ordinary theory.

For instance, take the example given in Todhunter's *Differential Calculus*, § 221,

$$\phi(x) = c + (x-a)^{\frac{3}{2}} + (x-a)^{\frac{4}{3}},$$

Here $$\phi'(x) = \tfrac{2}{3}(x-a)^{-\frac{1}{2}} + \tfrac{4}{3}(x-a)^{\frac{1}{3}},$$

which is infinite when $x = a$, and

$$\phi''(x) = -\tfrac{2}{3}(x-a)^{-\frac{3}{2}} + \tfrac{4}{9}(x-a)^{-\frac{2}{3}},$$

which is negative when $x = a$, and $\phi(a)$ is a minimum, as Todhunter shews, by pointing out that $\phi(a+h)$ and $\phi(a-h)$ are both greater than a.

A. E. STEINTHAL.

TRANSACTIONS OF SOCIETIES.

London Mathematical Society.

Thursday, February 10th, 1881.—S. Roberts, Esq., F.R.S., *President*, in the Chair. Mr. Wooster Woodruff Beman, Michigan University, was elected a Member; and Prof. Rowe, M.A., and Mr. J. Parker Smith, M.A., were admitted into the Society.

The following communications were made: "On some integrals expressible in terms of the first complete elliptic integral and of gamma functions," Mr. J. W. L. Glaisher, F.R.S.; "Some theorems of kinematics on a sphere," Mr. E. B. Elliott, M.A.; "Supplement on binomial biordinals," Sir J. Cockle, F.R.S.; "An application of conjugate functions (to the case of membranes)," Mr. E. J. Routh, F.R.S.; "Note on Abel's theorem," Mr. T. Craig.

Thursday, March, 10th., 1881, S. Roberts, Esq., F.R.S., *President*, in the Chair. Prof. Cayley read a paper on "The Equilibrium and Flexure of a Skew Surface." Mr. Tucker communicated portions of papers, viz. "An application of Elliptic Functions to the Nodal Cubic," Mr. R. A. Roberts; and "Note on Prof. C. S. Peirce's Probability Notation of 1867," Mr. Hugh McColl. Mr. J. W. L. Glaisher, F.R.S. (*Vice-President*), having taken the Chair, the *President* communicated the following direct analogue in space of the well known plane theorem "If we take an arbitrary point on each side of a triangle and describe a circle through each vertex and the two points on adjacent sides, the three circles meet in a point"; viz. if we take an arbitrary point on each edge of a tetrahedron and describe a sphere through each vertex and the three points on adjacent edges, the four spheres meet in a point. The analogue was used as a point of departure for the study of four spheres meeting in a point.

R. TUCKER, M.A., *Hon. Sec.*

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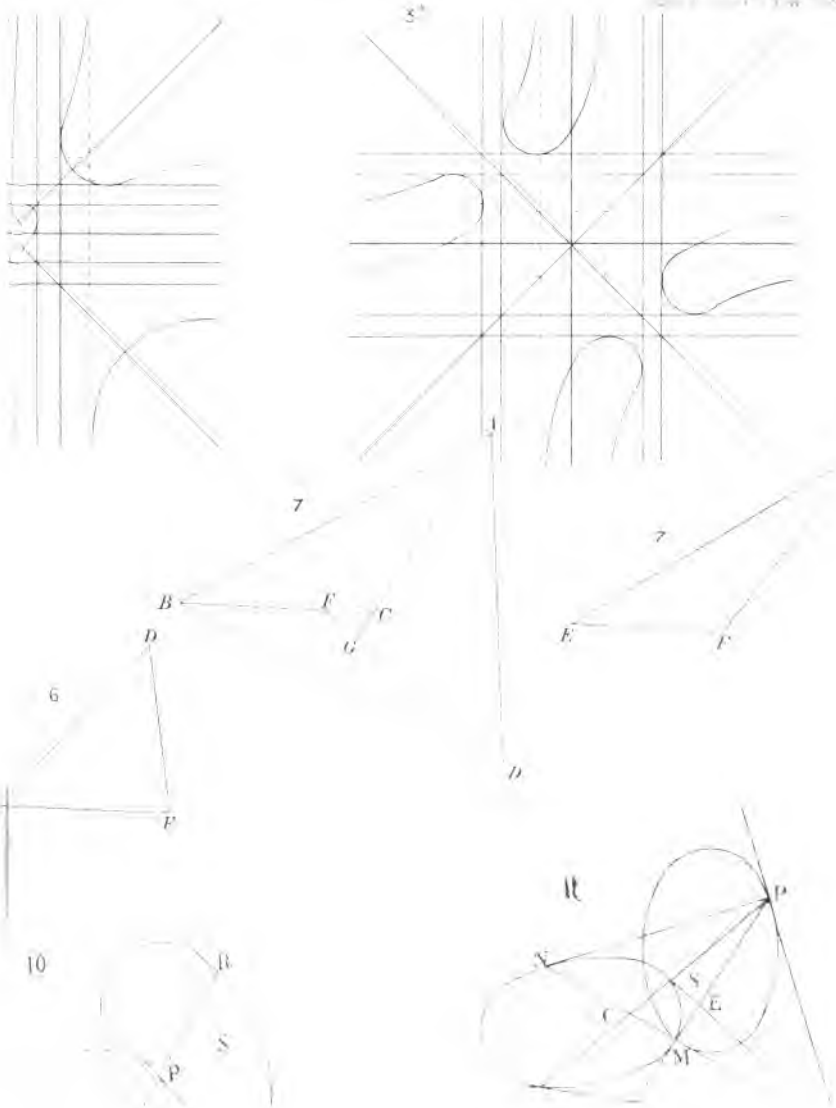

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Since x_0, y_0, ... and $x_0+\xi$, $y_0+\eta$, ... both satisfy the system of exact equations, we have

$$\left.\begin{aligned} A_1\xi + B_1\eta + \ldots &= 0 \\ A_2\xi + B_2\eta + \ldots &= 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots & \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (C).$$

Multiply equations (B) by p_1a_1, p_2a_2, &c. and add, then by p_1b_1, p_2b_2, &c. and add; we have, by means of equations (III),

$$\left.\begin{aligned} [paa]\,\xi + [pab]\,\eta + \ldots - [pau] - A_1\lambda_1 - A_2\lambda_2 - \ldots &= 0 \\ [pba]\,\xi + [pbb]\,\eta + \ldots - [pbu] - B_1\lambda_1 - B_2\lambda_2 - \ldots &= 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots & \end{aligned}\right\},$$

so that by comparison with (III) and (1)

$$\xi = \alpha_1 u_1 + \alpha_2 u_2 + \ldots,$$

where α_1, α_2, ... denote the same as before.

Now from equations (A), we have

$$\left.\begin{aligned} [pav] &= -A_1\lambda_1 - A_2\lambda_2 - \ldots \\ [pbv] &= -B_1\lambda_1 - B_2\lambda_2 - \ldots \\ \ldots\ldots\ldots\ldots & \end{aligned}\right\},$$

and from equations (B), we have by means of (III),

$$[pvv] + [pav]\,\xi + [pbv]\,\eta + \ldots = [puv],$$

i.e.

$$\left.\begin{aligned} [puv] = [pvv] &- \lambda_1[A_1\xi + B_1\eta + \ldots] \\ &- \lambda_2[A_2\xi + B_2\eta + \ldots] \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} = [pvv].$$

Also from equations (B)

$$\begin{aligned} [puu] &= [puv] + [pau]\,\xi + [pbu]\,\eta + \ldots \\ &= [pvv] + [pau]\,\xi + [pbu]\,\eta + \ldots. \end{aligned}$$

Now

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Thus, since $[puu] = m\varepsilon^2$, we should have

$$m\varepsilon^2 = [pvv] + \mu\varepsilon^2 - \varepsilon^2\,[A_1Q_{\mu+1} + A_2Q_{\mu+2} + \ldots + B_1R_{\mu+1} + B_2R_{\mu+2} + \ldots + \ldots\ldots\ldots\ldots\ldots\ldots].$$

Consider the equations

$$[paa]\,P_1 + [pba]\,P + \ldots + A_1P_{\mu+1} + A_2P_{\mu+2} + \ldots = \sigma_1,$$
$$[pba]\,P_1 + [pbb]\,P_2 + \ldots + B_1P_{\mu+1} + B_2P_{\mu+2} + \ldots = \sigma_2,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$A_1P_1 + A_2P_2 + \ldots = 0,$$
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Then, when $\sigma_1 = 1$, $\sigma_2 = 0$, &c., the P's are what we have denoted by Q's; if $\sigma_1 = 0$, $\sigma_2 = 1$, &c., then they are our R's and so on.

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Thus we see that
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For instance, take the example given in Todhunter's *Differential Calculus*, § 221,

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which is infinite when $x = a$, and

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London Mathematical Society.

Thursday, February 10th, 1881.—S. Roberts, Esq., F.R.S., *President*, in the Chair. Mr. Wooster Woodruff Beman, Michigan University, was elected a Member; and Prof. Rowe, M.A., and Mr. J. Parker Smith, M.A., were admitted into the Society.

The following communications were made: "On some integrals expressible in terms of the first complete elliptic integral and of gamma functions," Mr. J. W. L. Glaisher, F.R.S.; "Some theorems of kinematics on a sphere," Mr. E. B. Elliott, M.A.; "Supplement on binomial biordinals," Sir J. Cockle, F.R.S.; "An application of conjugate functions (to the case of membranes)," Mr. E. J. Routh, F.R.S.; "Note on Abel's theorem," Mr. T. Craig.

Thursday, March, 10th., 1881, S. Roberts, Esq., F.R.S., *President*, in the Chair. Prof. Cayley read a paper on "The Equilibrium and Flexure of a Skew Surface." Mr. Tucker communicated portions of papers, viz. "An application of Elliptic Functions to the Nodal Cubic," Mr. R. A. Roberts; and "Note on Prof. C. S. Peirce's Probability Notation of 1867," Mr. Hugh McColl. Mr. J. W. L. Glaisher, F.R.S. (*Vice-President*), having taken the Chair, the *President* communicated the following direct analogue in space of the well known plane theorem "If we take an arbitrary point on each side of a triangle and describe a circle through each vertex and the two points on adjacent sides, the three circles meet in a point"; viz. if we take an arbitrary point on each edge of a tetrahedron and describe a sphere through each vertex and the three points on adjacent edges, the four spheres meet in a point. The analogue was used as a point of departure for the study of four spheres meeting in a point.

R. TUCKER, M.A., *Hon. Sec.*

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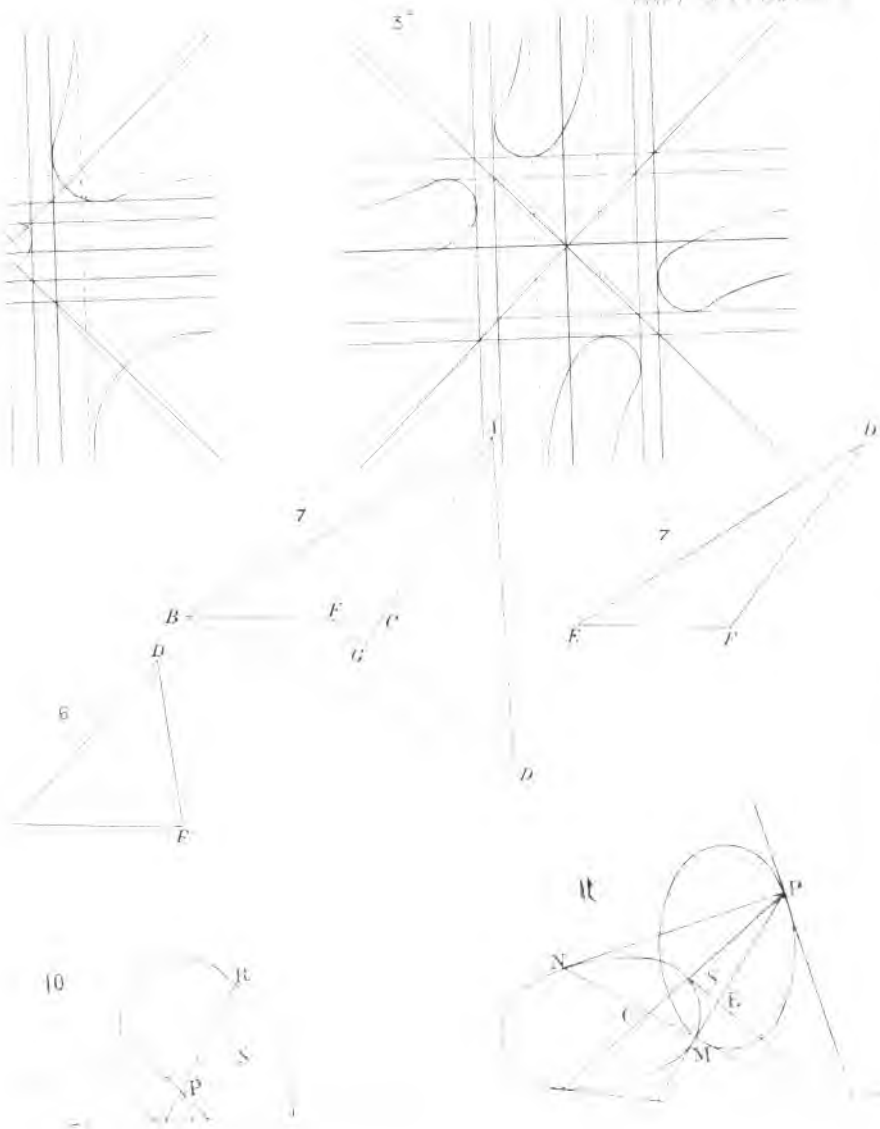

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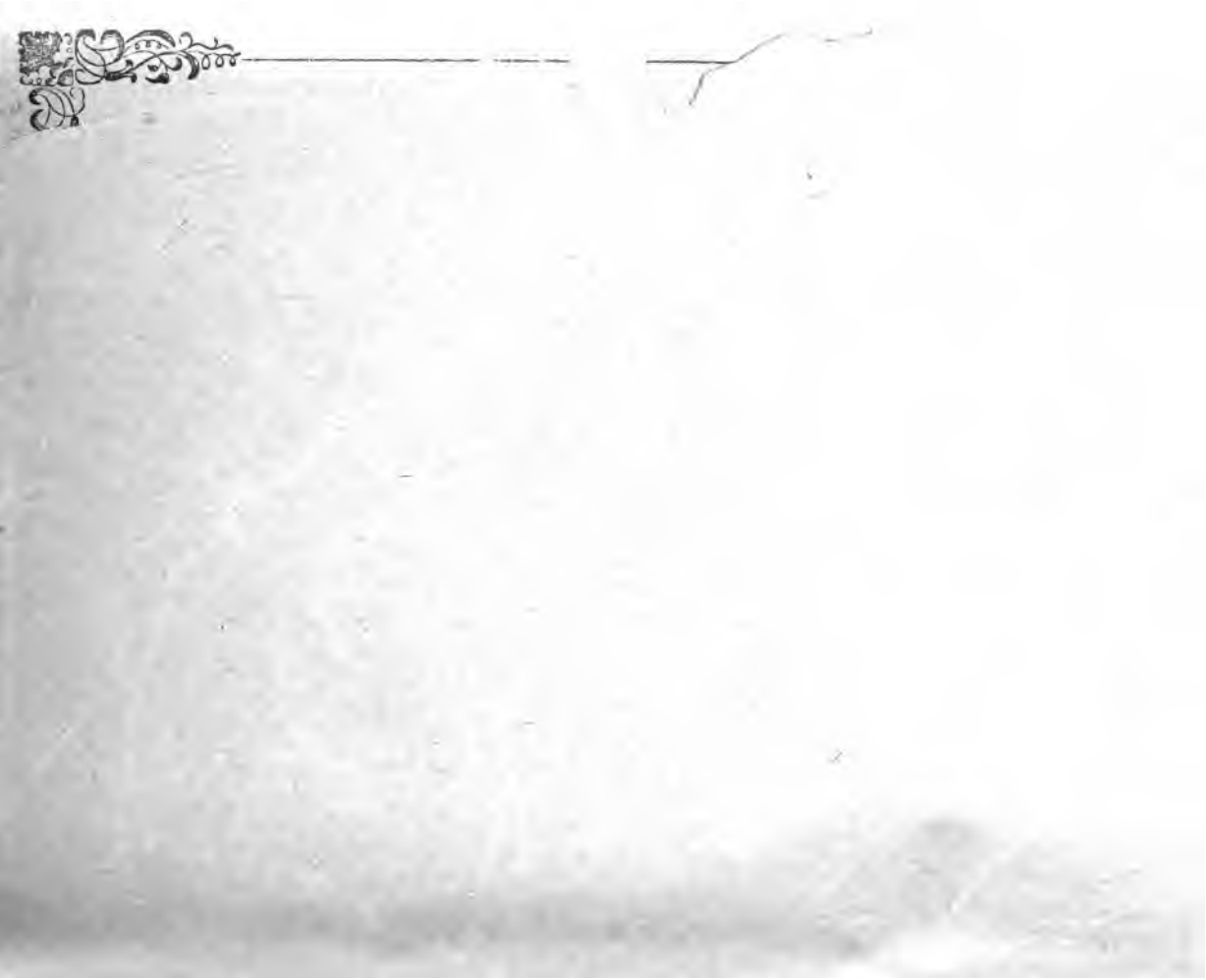

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